Jacobi

Disquisitiones Analyticae

DISQUISITIONES ANALYTICÆ

DE

FRACTIONIBUS SIMPLICIBUS.

AUCTORE

CAROLUS GUSTAVUS JACOBUS JACOBI

PHIL. DOCT.

BEROLINI, MDCCCXXV.

APUD F. A. HERBIG.

Sectio I.

Demonstratur theorema ab Ill. Lagrange sine demonstratione propositum.

1.

Mirum videri possit, et fortasse temerarium, si quis in materia inde a primis recentioris Analyseos temporibus a plurimis mathematicis tractata, quam igitur iure optimo decantatam dicere licet, vel novi quid velit afferre, vel ita rem attingere, ut ne acta egisse videatur. Iam vero fractionum simplicium theoriam ita fere decantatam esse vel inde patet, quod mathematici omnes, qui de serierum recurrentium theoria, omnes, qui de Calculi Integralis elementis egerunt, etiam de illis agere debuerunt. Sane nos quoque ista turba deterruisset, nisi casu in manus incidisset commentatio Ill. Lagrange, quae in Actis Academiae nostrae Berolinensis a. 1792—1793 legitur. Ibi enim, dum ille formulas quasdam in Actis eiusdem Academiae a. 1775 ab ipso exhibitas retractat, curiosam movit quaestionem de eiusmodi fractionum simplicium expressione investiganda, quae etsi denomitarum fractionum simplicium vel duo vel plures inter se aequales evadant, immutata maneret, ita ut ad speciem absurdi, quae istis casibus subnascitur, declinandam, non opus sit ad analyticam confugere transformationem. Ipse eiusmodi expressionem in medium profert, quam ut directa quadam methodo demonstrent, invitat geometras, cum ipse formulae propositae non addiderit demonstrationem. Unde in his quoque non ita omnia absoluta esse videbantur. Quod autem dico, hoc est.

2.

Propositam aliquam fractionem

$$\frac{fx}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\,.\,.\,(x-\alpha_n)},$$

designante fx functionem elementi x integram rationalem*) huiusmodi schematis

*) v. Eul. Introd. ad Anal. Infin. Lib. I. Cap. I. §. 8. 9

$$Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + P,$$

notum est in has resolvi posse fractiones simplices

$$\frac{f\alpha_1}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)} +$$
$$\frac{f\alpha_2}{(x-\alpha_2)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\ldots(\alpha_2-\alpha_n)} +$$
$$\frac{f\alpha_3}{(x-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\ldots(\alpha_3-\alpha_n)} +$$
$$\vdots$$
$$\frac{f\alpha_n}{(x-\alpha_n)(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\ldots(\alpha_n-\alpha_1)}. \text{ *)}$$

Quia posito denominatore

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n) = \varphi x,$$

facile patet, fore

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n) = \varphi'(\alpha_1) \text{ **)}$$
$$(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\ldots(\alpha_2-\alpha_n) = \varphi'(\alpha_2)$$
$$(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\ldots(\alpha_3-\alpha_n) = \varphi'(\alpha_3)$$
$$\vdots$$
$$(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\ldots(\alpha_n-\alpha_{n-1}) = \varphi'(\alpha_n),$$

fractiones illas simplices ita quoque scribere licet

$$\frac{f\alpha_1}{(x-\alpha_1)\varphi'(\alpha_1)} + \frac{f\alpha_2}{(x-\alpha_2)\varphi'(\alpha_2)} + \frac{f\alpha_3}{(x-\alpha_3)\varphi'(\alpha_3)} + \ldots + \frac{f\alpha_n}{(x-\alpha_n)\varphi'(\alpha_n)}$$

3.

Iam ubi quantitatum $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ aliquot aequales fiunt, expressionum $\varphi'(\alpha_1)$, $\varphi'(\alpha_2)$, $\varphi'(\alpha_3)$, .., $\varphi'(\alpha_n)$ totidem evanescunt, totidem fractionum simplicium

$$\frac{f\alpha_1}{(x-\alpha_1)\varphi'(\alpha_1)},\ \frac{f\alpha_2}{(x-\alpha_2)\varphi'(\alpha_2)},\ \frac{f\alpha_3}{(x-\alpha_3)\varphi'(\alpha_3)}, \ldots\ \frac{f\alpha_n}{(x-\alpha_n)\varphi'(\alpha_n)}$$

in infinitum abeunt. Scilicet ubi erit e. g.

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_m,$$

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*) Ubi fx ad altiorem quam $(n-1)^{tum}$ gradum ascendit, quo casu fractio

$$\frac{fx}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)}$$

spuria dici solet, i. e. functio rationalis ex integra et fracta conflata: fractiones illae simplices genuinam fractionem exprimunt in spuria illa latitantem.

**) Hic et in sequentibus, duce Ill. Lagrange, brevitatis causa ponimus $\frac{d\varphi x}{dx} = \varphi'(x)$, $\frac{d^2\varphi x}{dx^2} = \varphi'' x$, $\frac{d^3\varphi x}{dx^3} = \varphi''' x$, et in genere $\frac{d^n\varphi x}{dx^n} = \varphi^N x$; ubi tamen melius iudicabitur, veterem quoque designandi modum adhibebimus.

quorum valorem communem ponamus $= \alpha$: denominator φx factorem $(x - \alpha)^m$ continebit, quo factore $(x - \alpha)^m$ fractiones nasci constat simplices huiusmodi

$$\frac{a}{(x-\alpha)^m} + \frac{b}{(x-\alpha)^{m-1}} + \frac{c}{(x-\alpha)^{m-2}} + \cdot\cdot + \frac{p}{(x-\alpha)},$$

unde anterius fractionum simplicium schema omnino mutatur. Cum vero formulae alicuius schema suppositione quadam prorsus mutatur, per absurdi speciem id plerumque indicatur, sicuti hoc loco fractiones simplices

$$\frac{f\alpha_1}{(x-\alpha_1)\,\varphi'(\alpha_1)},\ \frac{f\alpha_2}{(x-\alpha_2)\,\varphi'(\alpha_2)},\ \frac{f\alpha_3}{(x-\alpha_3)\,\varphi'(\alpha_3)}, \cdot\cdot\ \frac{f\alpha_m}{(x-\alpha_m)\,\varphi'(\alpha_m)}$$

in infinitum abeunt; ita ut aut quaestio ea suppositione facta de integro retractanda sit, aut ad analyticam transformationem confugere debeamus, qua ista absurdi species declinetur.

Iam illi quidem numeratores a, b, c, . ., p facile consideratione sequenti inveniuntur. Sit enim $\varphi x = (x - \alpha)^m \psi x$, ita ut poni possit

$$\frac{fx}{\varphi x} = \frac{Fx}{\psi x} + \frac{a}{(x-\alpha)^m} + \frac{b}{(x-\alpha)^{m-1}} + \frac{c}{(x-\alpha)^{m-2}} + \cdot\cdot + \frac{p}{x-\alpha}.$$

Iam evoluta fractione proposita $\frac{fx}{\varphi x}$ ad dignitates ascendentes quantitatis $(x - \alpha)$, negativae, quae in illa evolutione inveniuntur, quantitatis $(x - \alpha)$ dignitates, hae ipsae evadunt fractiones simplices, in quas inquirimus:

$$\frac{a}{(x-\alpha)^m} + \frac{b}{(x-\alpha)^{m-1}} + \frac{c}{(x-\alpha)^{m-2}} + \cdot\cdot + \frac{p}{x-\alpha}.$$

Quia enim ψx factorem $x - \alpha$ non continere supponitur, in fractione $\frac{Fx}{\psi x}$ evoluta ad ascendentes quantitatis $x - \alpha$ dignitates, negativae eius dignitates inveniri non possunt.

Ut ipsam indicemus evolutionem, posito

$$\frac{fx}{\psi x} = \Pi x,$$

unde fractio proposita $\frac{fx}{\varphi x} = \frac{\Pi x}{(x-\alpha)^m}$, e theoremate Tayloriano fit

$$\Pi x = \Pi(\alpha + x - \alpha) = \Pi(\alpha) + \Pi'(\alpha)(x-\alpha) + \frac{\Pi''(x-\alpha)^2}{1.\ 2} + \frac{\Pi'''(x-\alpha)^3}{1.\ 2.\ 3.} + \text{etc.}$$

Hinc erit

$$\frac{fx}{\varphi x} = \frac{\Pi x}{(x-\alpha)^m} = \frac{\Pi(\alpha)}{(x-\alpha)^m} + \frac{\Pi'(\alpha)}{(x-\alpha)^{m-1}} + \frac{\Pi''(\alpha)}{1.2.(x-\alpha)^{m-2}} + \frac{\Pi'''(\alpha)}{1.2.3.(x-\alpha)^{m-3}} + \text{etc.},$$

unde statim quaesitas quantitates invenimus

$$a = \Pi\alpha,\ b = \Pi'(\alpha),\ c = \frac{\Pi''(\alpha)}{1.\ 2}, \cdot\cdot\ p = \frac{\Pi^{M-I}(\alpha)}{1.2.3..(m-1)};$$

Etsi non formulam, quam Ill. Lagrange voluit, methodum certe iam tradidimus, quae eadem

manet, quicunque sit numerus m, seu quotiescunque denominator φx factorem $(x-\alpha)$ contineat.

4.

Operae tamen pretium esse videbatur Analystis inquirere, quomodo hae formulae ex ipsa expressione §. 2 exhibita

$$\frac{f(\alpha_1)}{(x-\alpha_1)\,\varphi'(\alpha_1)} + \frac{f(\alpha_2)}{(x-\alpha_2)\,\varphi'(\alpha_2)} + \frac{f(\alpha_3)}{(x-\alpha_3)\,\varphi'(\alpha_3)} + \ldots + \frac{f(\alpha_n)}{(x-\alpha_n)\,\varphi'(\alpha_n)},$$

eo casu quo erit $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_m = \alpha$, analytica transformatione deducerentur. Quam quaestionem inter alios video suscepisse Ill. Malfatti in commentatione doctissima inscripta: *delle serie ricorrenti. V. Memorie di Matematica e Fisica della Societa Italiana*, Tom. III. pag. 571—663. Ibi ille, quas Ill. Lagrange in Actis Academiae nostrae a. 1775 sine demonstratione ea de re tradiderat formulas, falsas esse demonstravit, correctasque adstruxit, per calculos tamen valde prolixos et taediosos incedens. (plus XL illi paginas occupant.) Rem postea retractavit ipse Lagrange, iam a me citatus, in Actis Academiae nostrae Berolinensis a. 1792—93. Uterque eo artificio alibi etiam saepissime adhibito usus est, quod quantitates

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_m$$

non quidem aequales ab initio, sed quantitate infinite parva diversas statuerent. Quam denuo aggredi quaestionem operae pretium videbatur; quem ad finem duo antea proponamus lemmata, quae generaliori usui inservire possunt.

5.

Lemma I.

Posito $Fx = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_m)$, fractionem

$$\frac{1}{Fx} = \frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)},$$

in simplices resolutam vidimus fieri (§. 2)

$$\frac{1}{(x-\alpha_1)\,F'(\alpha_1)} + \frac{1}{(x-\alpha_2)\,F'(\alpha_2)} + \frac{1}{(x-\alpha_3)\,F'(\alpha_3)} + \ldots + \frac{1}{(x-\alpha_m)\,F'(\alpha_m)}.$$

Iam evoluta fractione

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_m)}$$

in seriem secundum descendentes elementi x dignitates procedentem, fit

$$\frac{1}{Fx} = \frac{1}{x^m} + \frac{\overset{1}{'C}}{x^{m+1}} + \frac{\overset{2}{'C}}{x^{m+2}} + \frac{\overset{3}{'C}}{x^{m+3}} + \text{etc.}$$

$$[\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ \alpha_m],$$

ubi per characteres

$$\overset{1}{'C}, \quad \overset{2}{'C}, \quad \overset{3}{'C}, \quad , \text{ etc.}$$

$$[\alpha_1, \quad \alpha_2, \quad \alpha_3, \; . \; . \; , \alpha_m]$$

more inter Analystas Germanos recepto combinationes designantur singulorum (i. summa), binorum, ternorum, etc. ex elementis

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \; . \; . \; , \alpha_m,$$

quae indice subscripto indicantur, ipsis elementorum admissis repetitionibus.

V. *Euler Introd. in Anal. Infinit. l. I. cap. XV. §.* 270.

Evolutis igitur etiam fractionibus simplicibus, in quas fractionem $\frac{1}{Fx}$ resolvimus, in seriem secundum descendentes elementi x dignitates procedentem, singularum dignitatum coëfficientes in utraque evolutione inter se comparando, sequentes eruimus aequationes satis memorabiles:

$$\frac{1}{F'(\alpha_1)} + \frac{1}{F'(\alpha_2)} + \frac{1}{F'(\alpha_3)} + \ldots + \frac{1}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1}{F'(\alpha_1)} + \frac{\alpha_2}{F'(\alpha_2)} + \frac{\alpha_3}{F'(\alpha_3)} + \ldots + \frac{\alpha_m}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1^2}{F'(\alpha_1)} + \frac{\alpha_2^2}{F'(\alpha_2)} + \frac{\alpha_3^2}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^2}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1^3}{F'(\alpha_1)} + \frac{\alpha_2^3}{F'(\alpha_2)} + \frac{\alpha_3^3}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^3}{F'(\alpha_m)} = 0$$

$$\ldots\ldots\ldots\ldots$$

$$\frac{\alpha_1^{m-2}}{F'(\alpha_1)} + \frac{\alpha_2^{m-2}}{F'(\alpha_2)} + \frac{\alpha_3^{m-2}}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^{m-2}}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1^{m-1}}{F'(\alpha_1)} + \frac{\alpha_2^{m-1}}{F'(\alpha_2)} + \frac{\alpha_3^{m-1}}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^{m-1}}{F'(\alpha_m)} = 1$$

$$\frac{\alpha_1^{m}}{F'(\alpha_1)} + \frac{\alpha_2^{m}}{F'(\alpha_2)} + \frac{\alpha_3^{m}}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^{m}}{F'(\alpha_m)} = \overset{1}{'C}$$

$$\frac{\alpha_1^{m+1}}{F'(\alpha_1)} + \frac{\alpha_2^{m+1}}{F'(\alpha_2)} + \frac{\alpha_3^{m+1}}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^{m+1}}{F'(\alpha_m)} = \overset{2}{'C}$$

$$\frac{\alpha_1^{m+2}}{F'(\alpha_1)} + \frac{\alpha_2^{m+2}}{F'(\alpha_2)} + \frac{\alpha_3^{m+2}}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^{m+2}}{F'(\alpha_m)} = \overset{3}{'C}$$

etc. etc.,

sive in universum

$$\frac{\alpha_1^{m+p}}{F'(\alpha_1)} + \frac{\alpha_2^{m+p}}{F'(\alpha_2)} + \frac{\alpha_3^{m+p}}{F'(\alpha_3)} + \ldots + \frac{\alpha_m^{m+p}}{F'(\alpha_m)} = \overset{p+1}{'C};$$

in quibus formulis characteres $\overset{1}{'C}$, $\overset{2}{'C}$, $\overset{3}{'C}$, et in universum $\overset{p+1}{'C}$ ad indicem communem

$$[\alpha_1, \ \alpha_2, \ \alpha_3, \ . \ . \ . \ \alpha_m]$$

referendi sunt.

6.

Lemma II.

Iam designante χx functionem aliquam elementi x, investigetur, quaenam evadat expressio

$$\frac{\chi\alpha_1}{F'(\alpha_1)} + \frac{\chi\alpha_2}{F'(\alpha_2)} + \frac{\chi\alpha_3}{F'(\alpha_3)} + . . + \frac{\chi\alpha_m}{F'(\alpha_m)}$$

posito $\alpha_1 = \alpha_2 = \alpha_3 = . . = \alpha_m$, quo casu singulas has fractiones in infinitum abire videmus.

Ponatur $\alpha_1 = \alpha + h_1$, $\alpha_2 = \alpha + h_2$, $\alpha_3 = a + h_3, . . ,$ $\alpha_m = \alpha + h_m$; atque $\Phi x = (x - h_1)(x - h_2)(x - h_3) . . (x - h_m)$; unde erit

$$\Phi(x - \alpha) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) . . (x - \alpha_m) = F(x).$$

Iam ex aequatione $Fx = \Phi(x - \alpha)$ sequitur $F'x = \Phi'(x - \alpha)$, unde

$$\begin{aligned} F'(\alpha_1) &= \Phi'(\alpha_1 - \alpha) = \Phi'(h_1) \\ F'(\alpha_2) &= \Phi'(\alpha_2 - \alpha) = \Phi'(h_2) \\ F'(\alpha_3) &= \Phi'(\alpha_3 - \alpha) = \Phi'(h_3) \\ & . \ . \ . \ . \ . \ . \ . \ . \ . \\ F'(\alpha_m) &= \Phi'(\alpha_m - \alpha) = \Phi'(h_m). \end{aligned}$$

Hinc expressio proposita

$$\frac{\chi\alpha_1}{F'(\alpha_1)} + \frac{\chi\alpha_2}{F'(\alpha_2)} + \frac{\chi\alpha_3}{F'(\alpha_3)} + . . + \frac{\chi\alpha_m}{F'(\alpha_m)}$$

in hanc abit

$$\frac{\chi(\alpha + h_1)}{\Phi'(h_1)} + \frac{\chi(\alpha + h_2)}{\Phi'(h_2)} + \frac{\chi(\alpha + h_3)}{\Phi'(h_3)} + . . + \frac{\chi(\alpha + h_m)}{\Phi'(h_m)}.$$

Quod evolutum secundum theorema Taylorianum ponatur =

$$A_0\,\chi\alpha + A_1\,\chi'(\alpha) + \frac{A_2\,\chi''(\alpha)}{1 \cdot 2} + \frac{A_3\,\chi'''(\alpha)}{1 \cdot 2 \cdot 3} + . . + \frac{A_{m-2}\,\chi^{M-II}(\alpha)}{1 \cdot 2 \, . . \, (m-2)}$$

$$+ \frac{A_{m-1}\,\chi^{M-I}(\alpha)}{1 \cdot 2 \, . . \, (m-1)} + \frac{A_m\,\chi^{M}(\alpha)}{1 \cdot 2 \, . . \, m} + \frac{A_{m+1}\,\chi^{M+I}(\alpha)}{1 \cdot 2 \, . . \, (m+1)} + \frac{A_{m+2}\,\chi^{M+II}(\alpha)}{1 \cdot 2 \, . . \, (m+2)} + \text{etc.};$$

unde erit

$$A_0 \;= \frac{1}{\Phi'(h_1)} + \frac{1}{\Phi'(h_2)} + \frac{1}{\Phi'(h_3)} + \ldots + \frac{1}{\Phi'(h_m)}$$

$$A_1 \;= \frac{h_1}{\Phi'(h_1)} + \frac{h_2}{\Phi'(h_2)} + \frac{h_3}{\Phi'(h_3)} + \ldots + \frac{h_m}{\Phi'(h_m)}$$

$$A_2 \;= \frac{h_2^2}{\Phi'(h_1)} + \frac{h_2^2}{\Phi'(h_2)} + \frac{h_3^2}{\Phi'(h_3)} + \ldots + \frac{h_m^2}{\Phi'(h_m)}$$

$$A_3 \;= \frac{h_1^3}{\Phi'(h_3)} + \frac{h_2^3}{\Phi'(h_2)} + \frac{h_3^3}{\Phi'(h_3)} + \ldots + \frac{h_m^3}{\Phi'(h_m)}$$

. . . .

$$A_{m-2} = \frac{h_1^{m-2}}{\Phi'(h_1)} + \frac{h_2^{m-2}}{\Phi'(h_2)} + \frac{h_3^{m-2}}{\Phi'(h_3)} + \ldots + \frac{h_m^{m-2}}{\Phi'(h_m)}$$

$$A_{m-1} = \frac{h_1^{m-1}}{\Phi'(h_1)} + \frac{h_2^{m-1}}{\Phi'(h_2)} + \frac{h_3^{m-1}}{\Phi'(h_3)} + \ldots + \frac{h_m^{m-1}}{\Phi'(h_m)}$$

$$A_m \;= \frac{h_1^{m}}{\Phi'(h_1)} + \frac{h_2^{m}}{\Phi'(h_2)} + \frac{h_3^{m}}{\Phi'(h_3)} + \ldots + \frac{h_m^{m}}{\Phi'(h_m)}$$

$$A_{m+1} = \frac{h_1^{m+1}}{\Phi'(h_1)} + \frac{h_2^{m+1}}{\Phi'(h_2)} + \frac{h_3^{m+1}}{\Phi'(h_3)} + \ldots + \frac{h_m^{m+1}}{\Phi'(h_m)}$$

$$A_{m+2} = \frac{h_1^{m+2}}{\Phi'(h_1)} + \frac{h_2^{m+2}}{\Phi'(h_2)} + \frac{h_3^{m+2}}{\Phi'(h_3)} + \ldots + \frac{h_m^{m+2}}{\Phi'(h_m)}$$

etc. etc.

Iam e lemmate I. (§. 5.) sequitur, ubi loco $\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_m$ ponitur $h_1, h_2, h_3, \ldots, h_m$, atque Φx loco Fx,

$$A_0 = 0,\; A_1 = 0,\; A_2 = 0,\; A_3 = 0, \ldots, A_{m-2} = 0,$$

$$A_{m-1} = 1,\; A_m = {}'\overset{1}{C},\; A_{m+1} = {}'\overset{2}{C},\; A_{m+2} = {}'\overset{3}{C}, \text{ etc. etc.},$$

characteribus $'\overset{1}{C}, {}'\overset{2}{C}, {}'\overset{3}{C}$, etc. relatis ad indicem communem:

$$[h_1,\; h_2,\; h_3,\; \ldots,\; h_m];$$

ita ut sit expressio nostra proposita

$$\frac{\chi\alpha_1}{F'(\alpha_1)} + \frac{\chi\alpha_2}{F'(\alpha_2)} + \frac{\chi\alpha_3}{F'(\alpha_3)} + \ldots + \frac{\chi\alpha_m}{F'(\alpha_m)} =$$

$$\frac{\overset{M-I}{\chi}(\alpha)}{1.2..(m-1)} + \frac{{}'\overset{1}{C}\overset{M}{\chi}(\alpha)}{1.2..m} + \frac{{}'\overset{2}{C}\overset{M+I}{\chi}(\alpha)}{1.2..(m+1)} + \frac{{}'\overset{3}{C}\overset{M+II}{\chi}}{1.2..(m+2)} + \text{etc.}$$

$$[h_1,\; h_2,\; h_3, \ldots h_m]$$

Posito $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_m = \alpha$, exsistit

$$h_1 = h_2 = h_3 = \ldots = h_m = 0,$$

ita ut etiam, ea suppositione facta,

$$'\overset{1}{C} = o, \quad '\overset{2}{C} = o, \quad '\overset{3}{C} = o, \quad \text{etc.};$$

$$[\, h_1, \; h_2, \; h_3, \; \ldots \; h_m \,]$$

unde expressio proposita fit

$$\frac{\overset{M-I}{\chi}(\alpha)}{1 \cdot 2 \ldots (m-1)} = \frac{d^{m-1} \chi(\alpha)}{1 \cdot 2 \ldots (m-1)\, d\alpha^{m-1}}.$$

7.

Iam ad propositam quaestionem redeamus. Quaesivimus enim, resoluta fractione

$$\frac{fx}{\varphi x} = \frac{fx}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)}$$

in simplices hasce

$$\frac{f\alpha_1}{(x-\alpha_1)\,\varphi'(\alpha_1)} + \frac{f\alpha_2}{(x-\alpha_2)\,\varphi'(\alpha_2)} + \frac{f\alpha_3}{(x-\alpha_3)\cdot\varphi'(\alpha_3)} + \ldots + \frac{f\alpha_n}{(x-\alpha_n)\,\varphi'(\alpha_n)};$$

quaenam evadant fractiones simplices, quae e denominatoris φx factoribus $(x-\alpha_1)$, $(x-\alpha_2)$, $(x-\alpha_3), \ldots, (x-\alpha_m)$ ortum ducunt, videlicet

$$\frac{f\alpha_1}{(x-\alpha_1)\,\varphi'(\alpha_1)} + \frac{f\alpha_2}{(x-\alpha_2)\,\varphi'(\alpha_2)} + \frac{f\alpha_3}{(x-\alpha_3)\,\varphi'(\alpha_3)} + \ldots + \frac{f\alpha_m}{(x-\alpha_m)\,\varphi'(\alpha_m)},$$

casu quo erit $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_m = \alpha$.

Posito $(x-\alpha_1)\,(x-\alpha_2)\,(x-\alpha_3)\ldots(x-\alpha_m) = Fx$, atque $\dfrac{fx}{(x-\alpha_{m+1})\,(x-\alpha_{m+2})\,(x-\alpha_{m+3})\ldots(x-\alpha_n)} = \Pi x$, fit fractio proposita

$$\frac{fx}{\varphi x} = \frac{\Pi x}{Fx},$$

unde etiam

$$\frac{\varphi x}{fx} = \frac{Fx}{\Pi x},$$

qua differentiata aequatione prodit

$$\frac{\varphi' x}{fx} + \varphi x \left(\frac{1}{fx}\right)' = \frac{F' x}{\Pi x} + Fx \left(\frac{1}{\Pi x}\right)',$$

unde in locum elementi x substitutis $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m$, quia ea substitutione facta evanescunt φx atque Fx, fit

$$\frac{\varphi'(\alpha_1)}{f\alpha_1} = \frac{F'(\alpha_1)}{\Pi\alpha_1},\; \frac{\varphi'(\alpha_2)}{f\alpha_2} = \frac{F'(\alpha_2)}{\Pi\alpha_2},\; \frac{\varphi'(\alpha_3)}{f\alpha_3} = \frac{F'(\alpha_3)}{\Pi\alpha_3}, \ldots, \frac{\varphi'(\alpha_m)}{f\alpha_m} = \frac{F'(\alpha_m)}{\Pi\alpha_m},$$

sive

$$\frac{f\alpha_1}{\varphi'(\alpha_1)} = \frac{\Pi\alpha_1}{F'(\alpha_1)},\; \frac{f\alpha_2}{\varphi'(\alpha_2)} = \frac{\Pi\alpha_2}{F'(\alpha_2)},\; \frac{f\alpha_3}{\varphi'(\alpha_3)} = \frac{\Pi\alpha_3}{F'(\alpha_3)}, \ldots, \frac{f\alpha_m}{\varphi'(\alpha_m)} = \frac{\Pi\alpha_m}{F'(\alpha_m)}.$$

Hinc

Hinc fit:

$$\frac{f\alpha_1}{(x-\alpha_1)\,\varphi'(\alpha_1)} + \frac{f\alpha_2}{(x-\alpha_2)\,\varphi'(\alpha_2)} + \frac{f\alpha_3}{(x-\alpha_3)\,\varphi'(\alpha_3)} + \ldots + \frac{f\alpha_m}{(x-\alpha_m)\,\varphi'(\alpha_m)} =$$

$$\frac{\Pi\alpha_1}{(x-\alpha_1)\,F'(\alpha_1)} + \frac{\Pi\alpha_2}{(x-\alpha_2)\,F'(\alpha_2)} + \frac{\Pi\alpha_3}{(x-\alpha_3)\,F'(\alpha_3)} + \ldots + \frac{\Pi\alpha_m}{(x-\alpha_m)\,F'(a_m)},$$

quae expressio e lemmate II. §. 6., posito $\alpha_1 = \alpha_2 = \alpha_3 \,.\,.= \alpha_m = \alpha$, fit

$$\frac{d^{m-1}\left(\frac{\Pi\alpha}{x-\alpha}\right)}{1\,.\,2\,.\,.\,(m-1)\,d\alpha^{m-1}};$$

loco $\chi\alpha$ enim ponendum erit $\frac{\Pi\alpha}{x-\alpha}$.

Facta differentiatione fit:

$$\frac{d^{m-1}\left(\frac{\Pi\alpha}{x-\alpha}\right)}{1\,.\,2\,.\,.\,(m-1)\,d\alpha^{m-1}} = \frac{\Pi\alpha}{(x-\alpha)^m} + \frac{\Pi'(\alpha)}{(x-\alpha)^{m-1}} + \frac{\Pi''(\alpha)}{1.2.(x-\alpha)^{m-2}} + \ldots + \frac{\Pi^{M-I}(\alpha)}{1.2\,.\,.(m\text{-}1)\,(x\text{-}\alpha)},$$

quod ipsum iam dedimus alia methodo inventum §. 3.

Haec transformatio analytica cum et ipsa digna quaestio videri potest, tum indicavit nobis, fractiones simplices, quae ex factore $(x-\alpha)^m$ ortum ducunt,

$$\frac{\Pi\alpha}{(x-\alpha)^m} + \frac{\Pi'(\alpha)}{(x-\alpha)^{m-1}} + \frac{\Pi''(\alpha)}{1.2.\,(x-\alpha)^{m-2}} + \ldots + \frac{\Pi^{M-I}(\alpha)}{1.2.(m-1)\;(x-\alpha)},$$

elegantissimum esse differentiale

$$\frac{d^{m-1}\left(\frac{\Pi\alpha}{x-\alpha}\right)}{1\,.\,2\,.\,.\,(m-1)\,d\alpha^{m-1}};$$

id quod methodus prius tradita non ita statim indicare videbatur. (v. tamen, quae in fine huius sectionis dedimus.)

8.

Initio huius commentationis §. 1. a nobis dictum est, Ill. Lagrange eiusmodi formulam tradidisse, quae, quotiescunque denominator φx factorem $(x-\alpha)$ contineat, nihil mutetur. Unde ex ea formula numerum m omnino evanuisse oportet.

Iam autem, designante in genere $\psi\omega$ seriem secundum elementi ω dignitates procedentem, coëfficientem dignitatis ω^p in serie illa $\psi\omega$ denotemus hic et in sequentibus per characterem

$$[\psi\omega]_{\omega^p};$$

ita ut e theoremate Tayloriano expressio

$$\frac{d^{m-1}\left(\frac{\Pi\alpha}{x-\alpha}\right)}{1.2\,..\,(m-1)\;d\alpha^{m-1}}$$

designari possit per characterem

$$\left[\frac{\Pi(\alpha+h)}{x-\alpha-h}\right]_{h^{m-1}}$$

Quia vero in genere

$$[\psi\omega]_{\omega^p} = [\omega^q\;\psi\omega]_{\omega^{p+q}},$$

erit etiam

$$\left[\frac{\Pi(\alpha+h)}{x-\alpha-h}\right]_{h^{m-1}} = \left[\frac{\Pi(\alpha+h)}{h^m(x-\alpha-h)}\right]_{h^{-1}}.$$

Iam vero erat (§. 6.)

$$\frac{fx}{\varphi x} = \frac{\Pi x}{(x-\alpha_1)(x-\alpha_2)\,..\,(x-\alpha_m}$$

unde, posito $\alpha_1 = \alpha_2 = .. = \alpha_m = \alpha$,

$$\frac{fx}{\varphi x} = \frac{\Pi x}{(x-\alpha)^m}.$$

In hac formula loco x substituamus $\alpha + h$; fit

$$\frac{f(\alpha+h)}{\varphi(\alpha+h)} = \frac{\Pi(\alpha+h)}{h^m},$$

unde

$$\frac{d^{m-1}\left(\frac{\Pi\alpha}{x-\alpha}\right)}{1.2\,..\,(m-1)\;d\alpha^{m-1}} = \left[\frac{\Pi(\alpha+h)}{h^m(x-\alpha-h)}\right]_{h^{-1}} = \left[\frac{f(\alpha+h)}{\varphi(\alpha+h)\,(x-\alpha-h)}\right]_{h^{-1}},$$

e qua formula numerum m prorsus evanuisse videmus. Nimirum invenimus, *fractione $\frac{fx}{\varphi x}$, cuius denominator φx factorem $x-\alpha$ continet, in fractiones simplices resoluta, eam fractionum simplicium partem, quae ex illo factore $x-\alpha$ ortum ducit, quotiescunque eum contineat denominator φx, esse*

$$\left[\frac{f(\alpha+h)}{\varphi(\alpha+h)\,(x-\alpha-h)}\right]_{h^{-1}}.$$

Hac expressione ad descendentes elementi x dignitates evoluta, fit terminus generalis

$$\frac{1}{x^{p+1}}\left[\frac{(\alpha+h)^p\;f(\alpha+h)}{\varphi(\alpha+h)}\right]_{h^{-1}}.$$

9.

Quod Ill. Lagrange loco citato proposuit theorema, mutatis mutandis, hoc est. Proposita serie

$$y_0,\ y_1,\ y_2,\ y_3,\ \ldots,\ y_p,\ y_{p+1},\ y_{p+2},\ \text{etc.},$$

data sit inter n + 1 quosque terminos seriei successivos aequatio:

$$a y_p + a_1 y_{p+1} + a_2 y_{p+2} \ldots + a_n y_{p+n} = 0,$$

unde videmus, seriem propositam e recurrentium numero esse, quippe cuius singuli termini ex evolutione fractionis alicuius huiusmodi

$$\frac{b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_1 x + b}{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a},$$

secundum descendentes elementi x dignitates facta, proveniunt; ita ut sit:

$$\frac{y_0}{x} + \frac{y_1}{x^2} + \frac{y_2}{x^3} + \frac{y_3}{x^4} + \ldots + \frac{y_{n-1}}{x^n} + \text{etc.} =$$

$$\frac{b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_1 x + b}{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a}.$$

Multiplicata enim serie

$$\frac{y_0}{x} + \frac{y_1}{x^2} + \frac{y_2}{x^3} + \frac{y_3}{x^4} + \ldots + \frac{y_{n-1}}{x^n} + \text{etc.}$$

per denominatorem

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a,$$

neque dignitates elementi x superiores $(n-1)^{ta}$ prodire videmus, et secundum legem illam, quae inter n + 1 terminos quosque successivos seriei

$$y_0,\ y_1,\ y_2,\ y_3,\ \ldots,\ y_p,\ y_{p+1},\ y_{p+2},\ \text{etc.}$$

intercedit, videlicet esse

$$a y_p + a_1 y_{p+1} + a_2 y_{p+2} + \ldots + a_n y_{p+n} = 0,$$

negativas elementi x dignitates evanescere omnes. Unde nansciscimur numeratorem

$$b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_1 x + b =$$

$$y_0 \left(a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1}\right) +$$

$$y_1 \left(a_2 + a_3 x + a_4 x^2 + \ldots + a_n x^{n-2}\right) +$$

$$y_2 \left(a_3 + a_4 x + a_5 x_2 + \ldots + a_n x^{n-3}\right) +$$

. . . .

$$y_{n-2}\,(a_{n-1} + a_n x) \; +$$
$$y_{n-1}\;\; a_n \,.$$

Iam posito

$$b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_1 x + b = fx,$$
$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a =$$
$$a_n (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n) = \varphi x,$$

fit $\dfrac{fx}{\varphi x} = \dfrac{y_0}{x} + \dfrac{y_1}{x^2} + \dfrac{y_2}{x^3} + \ldots + \dfrac{y_p}{x^{p+1}} +$ etc.

$$= \frac{f(\alpha_1)}{\varphi'(\alpha_1)(x-\alpha_1)} + \frac{f(\alpha_2)}{\varphi'(\alpha_2)(x-\alpha_2)} + \frac{f(\alpha_3)}{\varphi'(\alpha_3)(x-\alpha_3)} + \ldots + \frac{f(\alpha_n)}{\varphi'(\alpha_n)(x-a_n)};$$

ita ut terminus generalis y_p fiat

$$\frac{f(\alpha_1)\,\alpha_1^p}{\varphi'(\alpha_1)} + \frac{f(\alpha_2)\,\alpha_2^p}{\varphi'(\alpha_2)} + \frac{f(\alpha_3)\,\alpha_3^p}{\varphi'(\alpha_3)} + \ldots + \frac{f(\alpha_n)\,\alpha_n^p}{\varphi'(\alpha_n)}\,.$$

Iam ubi est

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_m = \alpha,$$

quaesivit Ill. Lagrange, quaenam evadat ea pars termini generalis y_p, quae e factoribus $x-\alpha_1$, $x-\alpha_2$, ..., $x-\alpha_m$ provenit, videlicet

$$\frac{f(\alpha_1)\,\alpha_1^p}{\varphi'(\alpha_1)} + \frac{f(\alpha_2)\,\alpha_2^p}{\varphi'(\alpha_2)} + \ldots + \frac{f(\alpha_m)\,\alpha_m^p}{\varphi'(\alpha_m)}\,.$$

Invenit, posito

$$P_0 = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \ldots + a_2 \alpha^2 + a_1 \alpha + a,$$
$$P_1 = a_n \alpha^{n-1} + a_{n-1} \alpha^{n-2} + \ldots + a_2 \alpha + a_1,$$
$$P_2 = a_n \alpha^{n-2} + a_{n-1} \alpha^{n-3} + \ldots + a_2,$$
$$\vdots$$
$$P_{n-1} = a_n \alpha + a_{n-1},$$
$$P_n = a_n,$$

ac denique

$$F(\alpha) = \left(P_1 y_0 + P_2 y_1 + P_3 y_2 + \ldots + P_n y_{n-1}\right)\alpha^p;$$

eam aequalem fore termino expressionis

$$\frac{F\alpha + h\dfrac{dF\alpha}{d\alpha} + h^2\dfrac{d^2F\alpha}{1.2.\,d\alpha^2} + h^3\dfrac{d^3F\alpha}{1.2.3.\,d\alpha^3} + \text{etc.}}{\dfrac{dP_0}{d\alpha} + h\dfrac{d^2P_0}{1.2.\,d\alpha^2} + h^2\dfrac{d^3P_0}{1.2.3.\,d\alpha^3} + h^3\dfrac{d^4P_0}{1.2.3.4.\,d\alpha^4} + \text{etc.}},$$

evolutae secundum dignitates elementi h ascendentes, termino dico illi, qui ab elemento h vacuus invenitur, qui nobis terminus denotatur per characterem

$$\left\{ \frac{F\alpha + h\frac{dF\alpha}{d\alpha} + h^2 \frac{d^2 F\alpha}{1.2.\,d\alpha^2} + h^3 \frac{d^3 F\alpha}{1.2.3.\,d\alpha^3} + \text{etc.}}{\frac{dP_0}{d\alpha} + h\frac{d^2 P_0}{1.2.d\alpha^2} + h^2 \frac{d^3 P_0}{1.2.3.\,d\alpha^3} + h^3 \frac{d^4 P_0}{1.2.3.4.\,d\alpha^4} + \text{etc.}} \right\} h^0.$$

Statim videmus e theoremate Tayloriano, fractionis numeratorem

$$F\alpha + h\frac{dF\alpha}{d\alpha} + h^2 \frac{d^2 F\alpha}{1.2.\,d\alpha^2} + h^3 \frac{d^3 F\alpha}{1.2.3.\,d\alpha^3} + \text{etc.}$$

esse $F(\alpha + h)$. Ex eodem theoremate sequitur etiam, denominatorem

$$\frac{dP_0}{d\alpha} + h\frac{d^2 P_0}{1.2.d\alpha^2} + h^2 \frac{d^3 P_0}{1.2.3.\,d\alpha^3} + h^3 \frac{d^4 P_0}{1.2.3.4.\,d\alpha^4} + \text{etc.}$$

esse $\dfrac{\varphi(\alpha + h) - \varphi\alpha}{h} = \dfrac{\varphi(\alpha + h)}{h}$. Est enim

$$P_0 = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \ldots + a_2 \alpha^2 + a_1 \alpha + a = \varphi(\alpha),$$

et quia posuimus α esse radicem aequationis

$$\varphi x = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a = 0,$$

erit $\varphi\alpha = 0$. Hinc expressio ab Ill. Lagrange exhibita in hanc abit

$$\left[\frac{h\,F(\alpha + h)}{\varphi(\alpha + h)}\right] h^0 = \left[\frac{F(\alpha + h)}{\varphi(\alpha + h)}\right] h^{-1}.$$

Ex aequatione autem:

$$\begin{aligned} b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_1 x + b = fx = \\ y_0 \quad (a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1}) + \\ y_1 \quad (a_2 + a_3 x + a_4 x^2 + \ldots + a_n x^{n-2}) + \\ y_2 \quad (a_3 + a_4 x + a_5 x^2 + \ldots + a_n x^{n-3}) + \\ \ldots\ldots\ldots\ldots \\ y_{n-2} (a_{n-1} + a_n x) + \\ y_{n-1}\, a_n \end{aligned}$$

sequitur $f\alpha =$

$$\begin{aligned} y_0 \quad (a_1 + a_2 \alpha + a_3 \alpha^2 + \ldots + a_n \alpha^{n-1}) + \\ y_1 \quad (a_2 + a_3 \alpha + a_4 \alpha^2 + \ldots + a_n \alpha^{n-2}) + \\ y_2 \quad (a_3 + a_4 \alpha + a_5 \alpha^2 + \ldots + a_n \alpha^{n-3}) + \\ \ldots\ldots\ldots\ldots \end{aligned}$$

$$y_{n-2}\,(a_{n-1} + a_n\,\alpha) \;+$$

$$y_{n-1}\;a_n$$

$$= y_0\,P_1 + y_1\,P_2 + y_2\,P_3 + \ldots + y_{n-2}\,P_{n-1} + y_{n-1}\,P_n,$$

unde $F(\alpha) =$

$$\left(y_0\,P_1 + y_1\,P_2 + y_2\,P_3 + \ldots + y_{n-2}\,P_{n-1} + y_{n-1}\,P_n\right)\alpha^p = f\alpha\,.\,\alpha^p.$$

Loco α posito $\alpha + h$, fit

$$F(\alpha + h) = f(\alpha + h)\,(\alpha + h)^p,$$

unde iam

$$\left[\frac{F(\alpha + h)}{\varphi(\alpha + h)}\right]h^{-1} = \left[\frac{f(\alpha + h)\,(\alpha + h)^p}{\varphi(\alpha + h)}\right]h^{-1},$$

quam eandem invenimus §. 7 formulam. Demonstratum igitur est, quod sine demonstratione dedit Ill. Lagrange theorema. Sub forma enim exhibitum paulo discrepante, idem esse vidimus atque illud a nobis probatum §. 8.

10.

Alia tamen via magis directa haec poterant inveniri. E nostro enim notationis modo, designante p numerum integrum positivum, est

$$\frac{1}{(x-\alpha)^p} = \left[\frac{1}{x-\alpha-h}\right]h^{p-1} = \left[\frac{1}{h^p}\cdot\frac{1}{x-\alpha-h}\right]h^{-1},$$

sive addito coëfficiente:

$$\frac{A_p}{(x-\alpha)^p} = \left[\frac{A_p}{h^p}\cdot\frac{1}{x-\alpha-h}\right]h^{-1}.$$

Iam sit $\dfrac{A_p}{(x-\alpha)^p}$ terminus generalis seriei, quae ad dignitates quantitatis $x-\alpha$ integras negativas procedit, et quam denotabimus per characterem

$$\Sigma\,\frac{A_p}{(x-\alpha)^p}.$$

Quibus positis, e formula

$$\frac{A_p}{(x-\alpha)^p} = \left[\frac{A_p}{h^p}\cdot\frac{1}{x-\alpha-h}\right]h^{-1}$$

statim sequitur:

$$\Sigma\,\frac{A_p}{(x-\alpha)^p} = \left[\Sigma\,\frac{A_p}{h^p}\cdot\frac{1}{x-\alpha-h}\right]h^{-1}.$$

Designante rursus

$$\Sigma A_q (x-\alpha)^q$$

seriem, quae secundum integras positivas quantitatis $x-\alpha$ dignitates procedit, videmus expressionem

$$\Sigma A_q h^q \cdot \frac{1}{x-\alpha-h},$$

fractione $\frac{1}{x-\alpha-h}$ semper in hisce ad ascendentes elementi h dignitates evoluta, nullas omnino continere dignitates elementi h negativas, neque igitur dignitatem h^{-1}. Hinc erit:

$$\left[\Sigma A_q h^q \cdot \frac{1}{x-\alpha-h}\right] h^{-1} = 0,$$

unde poni potest:

$$\Sigma \frac{A_p}{(x-\alpha)^p} = \left[\left(\Sigma \frac{A_p}{h^p} + \Sigma A_q h^q\right) \frac{1}{x-\alpha-h}\right] h^{-1}.$$

Posito:

$$\Sigma \frac{A_p}{(x-\alpha)^p} + \Sigma A_q (x-\alpha)^q = F(x-\alpha),$$

videmus $\Sigma \frac{A_p}{(x-\alpha)^p}$ eam partem functionis $F(x-\alpha)$ esse, quae negativas, $\Sigma A_q (x-\alpha)^q$ eam, quae positivas quantitatis $x-\alpha$ dignitates continet.

Loco $x-\alpha$ posito h, fit $\Sigma \frac{A_p}{h^p} + \Sigma A_q h^q = F(h)$.

Iam igitur ex aequatione:

$$\Sigma \frac{A_p}{(x-\alpha)^p} = \left[\left(\Sigma \frac{A_p}{h^p} + \Sigma A_q h^q\right) \frac{1}{x-\alpha-h}\right] h^{-1} = \left[\frac{F(h)}{x-\alpha-h}\right] h^{-1}$$

sequitur $\Sigma \frac{A_p}{(x-\alpha)^p}$, sive *eam partem functionis* $F(x-\alpha)$, *quae negativas tantum quantitatis* $x-\alpha$ *dignitates continet, aequalem esse expressioni*

$$\left[\frac{F(h)}{x-\alpha-h}\right]_h -1.$$

Vidimus autem §. 3, resoluta fractione aliqua proposita $\frac{fx}{\varphi x}$ in fractiones simplices, ubi denominator φx factorem $x-\alpha$ continet, fractiones simplices, quae ex eo factore proveniunt, eam partem fore fractionis $\frac{fx}{\varphi x}$, secundum ascendentes quantitatis $x-\alpha$ dignitates

evolutae, quae e negativis huius quantitatis $x-\alpha$ dignitatibus constat. Posito iam:

$$F(x-\alpha) = \frac{fx}{\varphi x},$$

unde, loco x substitnto $\alpha + h$,

$$F(h) = \frac{f(\alpha+h)}{\varphi(\alpha+h)},$$

sequitur e theoremate modo exhibito eam partem functionis $\frac{fx}{\varphi x}$, secundum quantitatis $x-\alpha$ dignitates ascendentes evolutae, quae e negativis huius quantitatis $x-\alpha$ dignitatibus constat, esse

$$\left[\frac{f(\alpha+h)}{\varphi(\alpha+h)} \cdot \frac{1}{x-\alpha-h}\right]_{h^{-1}},$$

unde etiam fractiones simplices, quae e factore $x-\alpha$ proveniunt, erunt

$$\left[\frac{f(\alpha+h)}{\varphi(\alpha+h)} \cdot \frac{1}{x-\alpha-h}\right]_{h^{-1}},$$

quam ipsam supra dedimus formulam, alia prorsus via inventam.

Generaliores, quae his superstrui possunt, disquisitiones alia exhibebimus occasione.

Sectio II.

Resolutio singularis cuiusdam problematis indeterminati.

11.

Fractionem aliquam propositam, cuius denominator in factores lineares, quos dicunt, resolutus est

$$(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n),$$

vidimus in fractiones simplices resolvi posse, quarum singuli denominatores hi ipsi sint factores

$$x-\alpha_1,\quad x-\alpha_2,\quad x-\alpha_3,\ \ldots,\quad x-\alpha_n.$$

Iam proposui mihi, datam fractionem in eiusmodi resolvere simpliciores, quarum denominatores singuli singula producta sint sive e binis, sive e ternis, sive e quaternis, etc. horum factorum:

$$x-\alpha_1,\quad x-\alpha_2,\quad x-\alpha_3,\ \ldots,\quad x-\alpha_n.$$

Resolvatur e. g. fractio proposita, ut a casu simplicissimo ordiamur, in fractiones simpliciores huiusmodi:

$A_{1,2}$

$$\frac{A_{1,2}}{(x-\alpha_1)(x-\alpha_2)} + \frac{A_{1,3}}{(x-\alpha_1)(x-\alpha_3)} + \ldots + \frac{A_{1,n}}{(x-\alpha_1)(x-\alpha_n)}$$
$$+ \frac{A_{2,3}}{(x-\alpha_2)(x-\alpha_3)} + \frac{A_{2,4}}{(x-\alpha_2)(x-\alpha_4)} + \ldots + \frac{A_{2,n}}{(x-\alpha_2)(x-\alpha_n)}$$
$$+ \frac{A_{3,4}}{(x-\alpha_3)(x-\alpha_4)} + \frac{A_{3,5}}{(x-\alpha_3)(x-\alpha_5)} + \ldots + \frac{A_{3,n}}{(x-\alpha_3)(x-\alpha_n)}$$
$$\cdots\cdots$$
$$+ \frac{A_{n-2,n-1}}{(x-\alpha_{n-2})(x-\alpha_{n-1})} + \frac{A_{n-2,n}}{(x-\alpha_{n-2})(x-\alpha_n)}$$
$$+ \frac{A_{n-1,n}}{(x-\alpha_{n-1})(x-\alpha_n)} .$$

Quae fractiones ubi sub eundem denominatorem

$$(x-\alpha_1)\ (x-\alpha_2)\ (x-\alpha_3)\ \ldots\ (x-\alpha_n)$$

colliguntur, numeratorem videmus $(n-2)^{\text{tum}}$ ordinem superare non posse, sive huiusmodi schematis fore:

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2} x^{n-2} .$$

Ubi igitur vicissim de fractione aliqua

$$\frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2} x^{n-2}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \ldots (x-\alpha_n)}$$

in dictas fractiones simpliciores resolvenda agitur, earum numeratores, quos elemento A adiectis indicibus designavimus, ita determinari debent, ut illae sub eundem denominatorem

$$(x-\alpha_1)\ (x-\alpha_2)\ (x-\alpha_3)\ \ldots\ (x-\alpha_n)$$

collectae, hunc ipsum nanciscantur numeratorem

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2} x^{n-2} .$$

Hinc singularum elementi x dignitatum coëfficientibus, quorum est numerus $n-1$, collatis, numeratores fractionum simpliciorum ita determinandos esse videmus, ut $n-1$ aequationibus seu conditionibus satisfaciant. Horum ipsorum vero fractionum simpliciorum numeratorum numerum esse videmus eundem atque combinationum binorum e n elementis, quarum est numerus $\frac{n(n-1)}{1 \cdot 2}$. Hinc problema a nobis propositum est indeterminatum. Quantitates enim quaesitae numero sunt $\frac{n(n-1)}{1 \cdot 2}$, sed $n-1$ tantum conditionibus satisfaciendum erit, unde in solutione problematis completa $\frac{n(n-1)}{1 \cdot 2} - (n-1) = \frac{(n-1)(n-2)}{1 \cdot 2}$ quantitates arbitrariae inveniantur necesse est. Quia vero certa non constat methodus, eleganter solvendi eiusmodi problemata indeterminata, dignum videbatur, in quod inquireretur, problema.

12.

In sequentibus, proposita aliqua functione elementorum $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, quae sit $f(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$, designabimus per characterem

$$\Sigma f(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$$

summam omnium eiusmodi expressionum $f(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$, quae omnibus modis inter se permutatis elementis $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ sive, quod idem est, eorum indicibus $1, 2, 3, \ldots, n$ eruuntur; in quo aggregato faciendo ab utroque cavendum est, ne quis omittatur terminus ac ne plus semel apponatur.

Iam analogiam secutus fractionum simplicium, de quibus sectione I. actum est, contemplatus sum expressionem

$$\Sigma \frac{\alpha_1^a \alpha_2^b + \alpha_1^b \alpha_2^a}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1,2}},$$

posito $M_{1,2} = (\alpha_1-\alpha_3)(\alpha_1-\alpha_4)\ldots(\alpha_1-\alpha_n) \times$
$(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)\ldots(\alpha_2-\alpha_n)$.

Huius expressionis colligamus fractiones omnes, quarum denominatores factorem $x-\alpha_1$ continent, quae erunt:

$$\frac{\alpha_1^a \alpha_2^b + \alpha_1^b \alpha_2^a}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1,2}} + \frac{\alpha_1^a \alpha_3^b + \alpha_1^b \alpha_3^a}{(x-\alpha_1)(x-\alpha_3)} \cdot \frac{1}{M_{1,3}}$$
$$+ \frac{\alpha_1^a \alpha_4^b + \alpha_1^b \alpha_4^a}{(x-\alpha_1)(x-\alpha_4)} \cdot \frac{1}{M_{1,4}} + \ldots + \frac{\alpha_1^a \alpha_n^b + \alpha_1^b \alpha_n^a}{(x-\alpha_1)(x-\alpha_n)} \cdot \frac{1}{M_{1,n}}.$$

Vix autem adnotari debet, expressiones $M_{1,3}, M_{1,4}, \ldots, M_{1,n}$ ex expressione $M_{1,2}$ demanare, elemento α_2 permutato cum elementis $\alpha_3, \alpha_4, \ldots, \alpha_n$.

Iam fractionibus

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)}, \frac{1}{(x-\alpha_1)(x-\alpha_3)}, \frac{1}{(x-\alpha_1)(x-\alpha_4)}, \ldots, \frac{1}{(x-\alpha_1)(x-\alpha_n)}$$

in simplices resolutis, fit:

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)} = \frac{1}{(x-\alpha_1)(\alpha_1-\alpha_2)} + \frac{1}{(x-\alpha_2)(\alpha_2-\alpha_1)}$$
$$\frac{1}{(x-\alpha_1)(x-\alpha_3)} = \frac{1}{(x-\alpha_1)(\alpha_1-\alpha_3)} + \frac{1}{(x-\alpha_3)(\alpha_3-\alpha_1)}$$
$$\frac{1}{(x-\alpha_1)(x-\alpha_4)} = \frac{1}{(x-\alpha_1)(\alpha_1-\alpha_4)} + \frac{1}{(x-\alpha_4)(\alpha_4-\alpha_1)}$$
$$\ldots\ldots\ldots$$
$$+ \frac{1}{(x-\alpha_1)(x-\alpha_n)} = \frac{1}{(x-\alpha_1)(\alpha_1-\alpha_n)} + \frac{1}{(x-\alpha_n)(\alpha_n-\alpha_1)}.$$

His valoribus fractionum

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)},\ \frac{1}{(x-\alpha_1)(x-\alpha_3)},\ \frac{1}{(x-\alpha_1)(x-\alpha_4)},\ \ldots,\ \frac{1}{(x-\alpha_1)(x-\alpha_n)}$$

in expressionem

$$\frac{\alpha_1^{\,a}\alpha_2^{\,b}+\alpha_1^{\,b}\alpha_2^{\,a}}{(x-\alpha_1)(x-\alpha_2)}\cdot\frac{1}{M_{1,2}}+\frac{\alpha_1^{\,a}\alpha_3^{\,b}+\alpha_1^{\,b}\alpha_3^{\,a}}{(x-\alpha_1)(x-\alpha_3)}\cdot\frac{1}{M_{1,3}}$$

$$+\frac{\alpha_1^{\,a}\alpha_4^{\,b}+\alpha_1^{\,b}\alpha_4^{\,a}}{(x-\alpha_1)(x-\alpha_4)}\cdot\frac{1}{M_{1,4}}+\ldots+\frac{\alpha_1^{\,a}\alpha_n^{\,b}+\alpha_1^{\,b}\alpha_n^{\,a}}{(x-\alpha_1)(x-\alpha_n)}\cdot\frac{1}{M_{1,n}}$$

substitutis, fractiones, quae denominatorem $x-\alpha_1$ habent, ubi pro expressionibus $M_{1,2}$, $M_{1,3}$, $M_{1,4}, \ldots, M_{1,n}$ earum valores ponuntur, fiunt:

$$\left\{\begin{array}{l}\dfrac{\alpha_1^{\,a}\alpha_2^{\,b}+\alpha_1^{\,b}\alpha_2^{\,a}}{(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)\ldots(\alpha_2-\alpha_n)}\\ +\dfrac{\alpha_1^{\,a}\alpha_3^{\,b}+\alpha_1^{\,b}\alpha_3^{\,a}}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)\ldots(\alpha_3-\alpha_n)}\\ +\dfrac{\alpha_1^{\,a}\alpha_4^{\,b}+\alpha_1^{\,b}\alpha_4^{\,a}}{(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)\ldots(\alpha_4-\alpha_n)}\\ \quad\cdots\cdots\\ +\dfrac{\alpha_1^{\,a}\alpha_n^{\,b}+\alpha_1^{\,b}\alpha_n^{\,a}}{(\alpha_n-\alpha_2)(\alpha_n-\alpha_3)\ldots(\alpha_n-\alpha_{n-1})}\end{array}\right\}\frac{1}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)\ldots(\alpha_1-\alpha_n)}$$

$$=$$

$$\left\{\begin{array}{l}\dfrac{\alpha_2^{\,b}}{(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)\ldots(\alpha_2-\alpha_n)}\\ +\dfrac{\alpha_3^{\,b}}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)\ldots(\alpha_3-\alpha_n)}\\ +\dfrac{\alpha_4^{\,b}}{(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)\ldots(\alpha_4-\alpha_n)}\\ \quad\cdots\cdots\\ +\dfrac{\alpha_n^{\,b}}{(\alpha_n-\alpha_2)(\alpha_n-\alpha_3)\ldots(\alpha_n-\alpha_{n-1})}\end{array}\right\}\frac{\alpha_1^{\,a}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}$$

$$+$$

$$\left\{\begin{array}{l}\dfrac{\alpha_2^{\,a}}{(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)\ldots(\alpha_2-\alpha_n)}\\ +\dfrac{\alpha_3^{\,a}}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)\ldots(\alpha_3-\alpha_n)}\\ +\dfrac{\alpha_4^{\,a}}{(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)\ldots(\alpha_4-\alpha_n)}\\ \quad\cdots\cdots\\ +\dfrac{\alpha_n^{\,a}}{(\alpha_n-\alpha_2)(\alpha_n-\alpha_3)\ldots(\alpha_n-\alpha_{n-1})}\end{array}\right\}\frac{\alpha_1^{\,b}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}$$

Posito $(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)\ldots(x-\alpha_n) = \varphi x$, invenitur:

$$\begin{aligned}
(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)\ldots(\alpha_2-\alpha_n) &= \varphi'(\alpha_2)\\
(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)\ldots(\alpha_3-\alpha_n) &= \varphi'(\alpha_3)\\
(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)\ldots(\alpha_4-\alpha_n) &= \varphi'(\alpha_4)\\
&\ldots\ldots\\
(\alpha_n-\alpha_2)(\alpha_n-\alpha_3)\ldots(\alpha_n-\alpha_{n-1}) &= \varphi'(\alpha_n).
\end{aligned}$$

Quibus in expressionem antecedentem substitutis, prodit:

$$\frac{\alpha_1^{\,a}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}\left(\frac{\alpha_2^{\,b}}{\varphi'(\alpha_2)}+\frac{\alpha_3^{\,b}}{\varphi'(\alpha_3)}+\frac{\alpha_4^{\,b}}{\varphi'(\alpha_4)}+\ldots+\frac{\alpha_n^{\,b}}{\varphi'(\alpha_n)}\right)$$
$$+\frac{\alpha_1^{\,b}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}\left(\frac{\alpha_2^{\,a}}{\varphi'(\alpha_2)}+\frac{\alpha_3^{\,a}}{\varphi'(\alpha_3)}+\frac{\alpha_4^{\,a}}{\varphi'(\alpha_4)}+\ldots+\frac{\alpha_n^{\,a}}{\varphi'(\alpha_n)}\right).$$

Omnino similia eruuntur, singulis fractionibus, quae expressione

$$\Sigma\,\frac{\alpha_1^{\,a}\alpha_2^{\,b}+\alpha_1^{\,b}\alpha_2^{\,a}}{(x-\alpha_1)(x-\alpha_2)}\cdot\frac{1}{M_{1,2}}$$

comprehenduntur, in fractiones simplices resolutis, pro iis fractionibus simplicibus, quae denominatores $x-\alpha_2$, $x-\alpha_3$, $x-\alpha_4$, ..., $x-\alpha_n$ habent.

13.

Iam ubi et a et b numeri integri positivi erunt, minores numero $n-2$, e lemmate I. sectionis I. (§. 5) sequitur:

$$\frac{\alpha_2^{\,b}}{\varphi'(\alpha_2)}+\frac{\alpha_3^{\,b}}{\varphi'(\alpha_3)}+\frac{\alpha_4^{\,b}}{\varphi'(\alpha_4)}+\ldots+\frac{\alpha_n^{\,b}}{\varphi'(\alpha_n)}=0$$
$$\frac{\alpha_2^{\,a}}{\varphi'(\alpha_2)}+\frac{\alpha_3^{\,a}}{\varphi'(\alpha_3)}+\frac{\alpha_4^{\,a}}{\varphi'(\alpha_4)}+\ldots+\frac{\alpha_n^{\,a}}{\varphi'(\alpha_n)}=0.$$

Eo igitur casu invenimus, fractiones simplices, quae $x-\alpha_1$ denominatorem habent, i. e.

$$\frac{\alpha_1^{\,a}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}\left[\frac{\alpha_2^{\,b}}{\varphi'(\alpha_2)}+\frac{\alpha_3^{\,b}}{\varphi'(\alpha_3)}+\frac{\alpha_4^{\,b}}{\varphi'(\alpha_4)}+\ldots+\frac{\alpha_n^{\,b}}{\varphi'(\alpha_n)}\right]$$
$$+\frac{\alpha_1^{\,b}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}\left[\frac{\alpha_2^{\,a}}{\varphi'(\alpha_2)}+\frac{\alpha_3^{\,a}}{\varphi'(\alpha_3)}+\frac{\alpha_4^{\,a}}{\varphi'(\alpha_4)}+\ldots+\frac{\alpha_n^{\,a}}{\varphi'(\alpha_n)}\right]$$

evanescere. Eodem modo etiam, quae $x-\alpha_2$, $x-\alpha_3$, $x-\alpha_4$, ..., $x-\alpha_n$ denominatores habent, evanescunt. Ubi igitur p et q numeri integri positivi sunt, minores numero $n-2$, fit

$$\Sigma\,\frac{\alpha_1^{\,p}\alpha_2^{\,q}+\alpha_1^{\,q}\alpha_2^{\,p}}{(x-\alpha_1)(x-\alpha_2)}\cdot\frac{1}{M_{1,2}}=0.$$

Hac enim expressione

$$\Sigma\,\frac{\alpha_1^{\,p}\alpha_2^{\,q}+\alpha_1^{\,q}\alpha_2^{\,p}}{(x-\alpha_1)(x-\alpha_2)}\cdot\frac{1}{M_{1,2}}$$

in fractiones simplices resoluta, cum singulae istae evanescant fractiones simplices, et ipsa evanescat necesse est.

Ubi vero erit quidem $a < n-2$, sed $b = n-2$, fit quidem ex eodem lemmate I.:

$$\frac{\alpha_2^a}{\varphi'(\alpha_2)} + \frac{\alpha_3^a}{\varphi'(\alpha_3)} + \frac{\alpha_4^a}{\varphi'(\alpha_4)} + \cdot\cdot + \frac{\alpha_n^a}{\varphi'(\alpha_n)} = 0,$$

$$\text{sed } \frac{\alpha_2^b}{\varphi'(\alpha_2)} + \frac{\alpha_3^b}{\varphi'(\alpha_3)} + \frac{\alpha_4^b}{\varphi'(\alpha_4)} + \cdot\cdot + \frac{\alpha_n^b}{\varphi'(\alpha_n)} =$$

$$\frac{\alpha_2^{n-2}}{\varphi'(\alpha_2)} + \frac{\alpha_3^{n-2}}{\varphi'(\alpha_3)} + \frac{\alpha_4^{n-2}}{\varphi'(\alpha_4)} + \cdot\cdot + \frac{\alpha_n^{n-2}}{\varphi'(\alpha_n)} = 1.$$

Hinc fractiones simplices, quae $x - \alpha_1$ denominatorem habent, et quas vidimus esse

$$\frac{\alpha_1^a}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdot\cdot(\alpha_1-\alpha_n)}\left(\frac{\alpha_2^b}{\varphi'(\alpha_2)} + \frac{\alpha_3^b}{\varphi'(\alpha_3)} + \frac{\alpha_4^b}{\varphi'(\alpha_4)} + \cdot\cdot + \frac{\alpha_n^b}{\varphi'(\alpha_n)}\right) +$$

$$\frac{\alpha_1^b}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdot\cdot(\alpha_1-\alpha_n)}\left(\frac{\alpha_2^a}{\varphi'(\alpha_2)} + \frac{\alpha_3^a}{\varphi'(\alpha_3)} + \frac{\alpha_4^a}{\varphi'(\alpha_4)} + \cdot\cdot + \frac{\alpha_n^a}{\varphi'(\alpha_n)}\right),$$

simpliciter fiunt

$$\frac{\alpha_1^a}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdot\cdot(\alpha_1-\alpha_n)}.$$

Ubi vero etiam $a = n-2$, fiunt

$$\frac{2\,\alpha_1^a}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdot\cdot(\alpha_1-\alpha_n)}.$$

Eodem modo fractiones simplices, quae $x-\alpha_2$, $x-\alpha_3$, $x-\alpha_4$, . . , $x-\alpha_n$ denominatores habent, fiunt, ubi $a < n-2$,

$$\frac{\alpha_2^a}{(x-\alpha_2)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\cdot\cdot(\alpha_2-\alpha_n)},$$

$$\frac{\alpha_3^a}{(x-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\cdot\cdot(\alpha_3-\alpha_n)},$$

$$\frac{\alpha_4^a}{(x-\alpha_4)(\alpha_4-\alpha_1)(\alpha_4-\alpha_2)\cdot\cdot(\alpha_4-\alpha_n)},$$

. . . .

$$\frac{\alpha_n^a}{(x-\alpha_n)(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\cdot\cdot(\alpha_n-\alpha_{n-1})}.$$

Ubi vero $a = n-2$, fiunt dupla harum expressionum.

Unde videmus, siquidem erit $a < n-2$, $b = n-2$, fore:

$$\Sigma\,\frac{\alpha_1^a\alpha_2^b + \alpha_1^b\alpha_2^a}{(x-\alpha_1)(x-\alpha_2)}\cdot\frac{1}{M_{1,2}} =$$

$$\Sigma \frac{\alpha_1^{a}\alpha_2^{n-2} + \alpha_1^{n-2}\alpha_2^{a}}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1/2}} =$$

$$\frac{\alpha_1^{a}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\,.\,.\,(\alpha_1-\alpha_n)}$$

$$+ \frac{\alpha_2^{a}}{(x-\alpha_2)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\,.\,.\,(\alpha_2-\alpha_n)}$$

$$+ \frac{\alpha_3^{a}}{(x-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\,.\,.\,(\alpha_3-\alpha_n)}$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$+ \frac{\alpha_n^{a}}{(x-\alpha_n)(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\,.\,.\,(\alpha_n-\alpha_{n-1})}\,.$$

At sectione I. §. 2 vidimus, esse etiam:

$$\frac{x^{a}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\,.\,.\,(x-\alpha_{n-1})} =$$

$$\frac{\alpha_1^{a}}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\,.\,.\,(\alpha_1-\alpha_n)}$$

$$+ \frac{\alpha_2^{a}}{(x-\alpha_2)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\,.\,.\,(\alpha_2-\alpha_n)}$$

$$+ \frac{\alpha_3^{a}}{(x-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\,.\,.\,(\alpha_3-\alpha_n)}$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$+ \frac{\alpha_n^{a}}{(x-\alpha_n)(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\,.\,.\,(\alpha_n-\alpha_{n-1})}\,;$$

unde colligimus:

$$\Sigma \frac{\alpha_1^{a}\alpha_2^{n-2} + \alpha_1^{n-2}\alpha_2^{a}}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1/2}} = \frac{x^{a}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\,.\,.\,(x-\alpha_n)}\;:$$

Ubi vero etiam $a = n-2$, fit:

$$\Sigma \frac{\alpha_1^{n-2}\alpha_2^{n-2} + \alpha_1^{n-2}\alpha_2^{n-2}}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1/2}} = 2\,\Sigma \frac{\alpha_1^{n-2}\,\alpha_2^{n-2}}{(x-\alpha_1)(x-\alpha_2)}$$

$$= \frac{2\,x^{a}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\,.\,.\,(x-\alpha_n)}\,,$$

sive:

$$\Sigma \frac{\alpha_1^{n-2}\,\alpha_2^{n-2}}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1/2}} = \frac{x^{n-2}}{(x-\alpha_1)(x-\alpha_2)\,.\,.\,(x-\alpha_n)}\,.$$

14.

Loco expressionis

$$\Sigma \frac{\alpha_1^a \alpha_2^{n-2} + \alpha_1^{n-2} \alpha_2^a}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}}$$

simplicius etiam scribere licet

$$\Sigma \frac{\alpha_1^a \alpha_2^{n-2}}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}};$$

ex hac enim expressione

$$\frac{\alpha_1^a \alpha_2^{n-2}}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}}$$

permutatione indicum 1, 2 nascitur altera

$$\frac{\alpha_1^{n-2} \alpha_2^a}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}}.$$

Hinc, quia signo Σ cunctas amplexi sumus permutationes, expressio

$$\Sigma \frac{\alpha_1^a \alpha_2^{n-2} + \alpha_1^{n-2} \alpha_2^a}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}}$$

alios non continebit terminos, atque illa

$$\Sigma \frac{\alpha_1^a \alpha_2^{n-2}}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}}.$$

Videmus igitur, ubi erit $\alpha = 0, 1, 2, 3, \ldots, n-2$, fore:

$$\frac{x^a}{(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)} = \Sigma \frac{\alpha_1^a \alpha_2^{n-2}}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}}$$

Ubi igitur fractioni propositae assignamus numeratorem

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{n-2} x^{n-2},$$

fit

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{n-2} x^{n-2}}{(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n)} =$$

$$\Sigma \frac{(a_0 + a_1 \alpha_1 + a_2 \alpha_1^2 + \ldots + a_{n-2} \alpha_1^{n-2})\, \alpha_2^{n-2}}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}}:$$

His autem addere licet tot expressiones huiusmodi

$$\Sigma \frac{\alpha_1^p \alpha_2^q + \alpha_1^q \alpha_2^p}{(x - \alpha_1)(x - \alpha_2)} \cdot \frac{1}{M_{1,2}},$$

singulas per quantitatem arbitrariam multiplicatas, quot modis terminus

$$\alpha_1^p \alpha_2^q + \alpha_1^q \alpha_2^p$$

variari potest, dum et p et q minores sunt numero n—2. Tum enim eiusmodi expressiones

$$\Sigma \frac{\alpha_1^p \alpha_2^q + \alpha_1^q \alpha_2^p}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1,2}}$$

vidimus evanescere. Apparet autem cunctorum eiusmodi terminorum exponentes effingere combinationes binorum ex elementis

$$0,\ 1,\ 2,\ 3,\ \ldots,\ n-3,$$

ipsis admissis elementorum repetitionibus; quorum elementorum cum sit numerus $n-2$, harum complexionum erit $\frac{(n-2)(n-1)}{1\,.\,2}$. Tot igitur modis terminus $\alpha_1^p \alpha_2^q + \alpha_1^q \alpha_2^p$ variari potest; tot quantitates arbitrariae in formula indicata reperiuntur; unde completam dedimus problematis resolutionem; tot enim, ut sit completa, requirebantur (v. §. 11).

15.

E theoremate a nobis exhibito:

$$\frac{x^a}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)} = \Sigma \frac{\alpha_1^a \alpha_2^{n-2}}{(x-\alpha_1)(x-\alpha_2)} \cdot \frac{1}{M_{1,2}},$$

posito $a = 0,\ 1,\ 2,\ 3,\ \ldots,\ n-2$, formulae emanant omnino similes iis, quas lemmate I. §. 5 dedimus. Utraque enim aequationis parte secundum descendentes elementi x dignitates evoluta, singularum dignitatum collatis exponentibus, eruimus, ubi $a < n-2$,

$$\Sigma \frac{\alpha_1^a \alpha_2^{n-2}}{M_{1,2}} = 0;$$

videmus enim

$$\Sigma \frac{\alpha_1^a \alpha_2^{n-2}}{M_{1,2}}$$

coëfficientem esse dignitatis $\frac{1}{x^2}$, quae in evoluta fractione

$$\frac{x^a}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)}$$

omnino non invenitur, dum $a < n-2$. Ubi vero $a = n-2$, videmus in evoluta fractione

$$\frac{x^{n-2}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)}$$

terminum $\frac{1}{x^2}$ inveniri, unde posito $a = n-2$, fit

$$\Sigma \frac{\alpha_1^{n-2} \alpha_2^{n-2}}{M_{1,2}} = 1.$$

Horum duorum lemmatum ope eadem via, quam et in antecedentibus ingressi sumus, ad sequens pervenitur theorema.

Posito

Posito
$$M_{1,2,3} = (\alpha_1 - \alpha_4)(\alpha_1 - \alpha_5)\ldots(\alpha_1 - \alpha_n) \times$$
$$(\alpha_2 - \alpha_4)(\alpha_2 - \alpha_5)\ldots(\alpha_2 - \alpha_n) \times$$
$$(\alpha_3 - \alpha_4)(\alpha_3 - \alpha_5)\ldots(\alpha_3 - \alpha_n),$$
fit

$$1)\quad \Sigma \frac{\alpha_1^p\, \alpha_2^q\, \alpha_3^r}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)} \cdot \frac{1}{M_{1,2,3}} = 0,$$

numerorum p, q, r, qui sunt integri positivi, duobus non attingentibus numerum $n - 3$, reliquo eundem non superante.

$$2)\quad \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-3} x^{n-3}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)} =$$
$$\Sigma \frac{(a_0 + a_1\alpha_1 + a_2\alpha_1^2 + \ldots + a_{n-3}\alpha_1^{n-3})\,\alpha_2^{n-3}\,\alpha_3^{n-3}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)} \cdot \frac{1}{M_{1,2,3}}.$$

Non est meum, per calculos prolixos terrorem incutere lectori, eodemque repetito negotio plurimas paginas implere. Hinc via tantummodo indicata, accuratiorem omisi demonstrationem, quam qui tentaverit, eandem esse videbit, quam et in antecedentibus dedimus. Statim etiam generale theorema annectam.

Theorema.

Posito
$$M_{1,2,3,\ldots,k} = (\alpha_1 - \alpha_{k+1})(\alpha_1 - \alpha_{k+2})\ldots(\alpha_1 - \alpha_n) \times$$
$$(\alpha_2 - \alpha_{k+1})(\alpha_2 - \alpha_{k+2})\ldots(\alpha_2 - \alpha_n) \times$$
$$(\alpha_3 - \alpha_{k+1})(\alpha_3 - \alpha_{k+2})\ldots(\alpha_3 - \alpha_n) \times$$
$$(\alpha_4 - \alpha_{k+1})(\alpha_4 - \alpha_{k+2})\ldots(\alpha_4 - \alpha_n) \times$$
$$\ldots\ldots\ldots\ldots$$
$$(\alpha_k - \alpha_{k+1})(\alpha_k - \alpha_{k+2})\ldots(\alpha_k - \alpha_n),$$

fit
$$1)\quad \Sigma \frac{\alpha_1^{p_1}\,\alpha_2^{p_2}\,\alpha_3^{p_3}\ldots\alpha_k^{p_k}}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)} \cdot \frac{1}{M_{1,2,\ldots,k}} = 0,$$

duobus e numeris p_1, p_2, p_3, ..., p_k, qui sunt integri positivi, non attingentibus numerum $n-k$, reliquis eundem non superantibus.

$$2)\quad \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-k} x^{n-k}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)} =$$
$$\Sigma \frac{(a_0 + a_1\alpha_1 + a_2\alpha_1^2 + \ldots + a_{n-k}\alpha_1^{n-k})\,\alpha_2^{n-k}\,\alpha_3^{n-k}\ldots\alpha_k^{n-k}}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)} \cdot \frac{1}{M_{1,2,\ldots,k}}.$$

Huic formulae tot expressiones

$$\Sigma \frac{\alpha_1^{p_1} \alpha_2^{p_2} \alpha_3^{p_3} \ldots \alpha_k^{p_k}}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)} \cdot \frac{1}{M_{1,2,\ldots,k}}$$

addere licet, singulas per quantitatem arbitrariam multiplicatas, quot modis terminus

$$\alpha_1^{p_1} \alpha_2^{p_2} \alpha_3^{p_3} \ldots \alpha_k^{p_k}$$

variari potest, dum e numeris p_1, p_2, p_3, ..., p_k duo minores sunt numero $n-k$, reliqui eundem non superant. Iis casibus enim omnes eiusmodi expressiones

$$\Sigma \frac{\alpha_1^{p_1} \alpha_2^{p_2} \alpha_3^{p_3} \ldots \alpha_k^{p_k}}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)} \cdot \frac{1}{M_{1,2,\ldots,k}}$$

diximus evanescere.

Corollarium.

16.

Si omnes numeri p_1, p_2, p_3, .., p_k numerum $n-k$ attingere possent, terminorum

$$\alpha_1^{p_1} \alpha_2^{p_2} \alpha_3^{p_3} \ldots \alpha_k^{p_k}$$

exponentes effingerent combinationes k^{norum} ex elementis

$$0, 1, 2, \ldots, n-k,$$

ipsis elementorum admissis repetitionibus. Quorum cum numerus sit $n-k+1$, complexionum numerus foret

$$\frac{(n-k+1)(n-k+2)(n-k+3)\ldots(n)}{1 \cdot 2 \cdot 3 \ldots k}.$$

De quo numero detrahendus est numerus eorum casuum, quibus $k-1$ e numeris p_1, p_2, p_3, .., p_k sive omnes k aequales erunt numero $n-k$. Ab his enim casibus abstinendum est. Hic vero numerus aperte erit $n-k+1$. Hinc termini

$$\alpha_1^{p_1} \alpha_2^{p_2} \alpha_3^{p_3} \ldots \alpha_n^{p_n},$$

duobus e numeris p_1, p_2, p_3, ..., p_k non attingentibus numerum $n-k$, reliquis eundem non superantibus,

$$\frac{(n-k+1)(n-k+2)(n-k+3)\ldots(n)}{1 \cdot 2 \cdot 3 \ldots k} - (n-k+1)$$

modis variari possunt. Totidem igitur quantitates arbitrariae in formula indicata inveniuntur. Fractiones autem simpliciores, quarum denominatores sunt eiusmodi producta $(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)$, eodem sunt numero atque combinationes k^{norum} ex elementis

$$1, 2, 3, \ldots, n,$$

id est numero

$$\frac{n\,(n-1)\,(n-2)\,.\,.\,(n-k+1)}{1\,.\,2\,.\,3\,\,.\,.\,\,k}.$$

Iam illae ita determinatos numeratores habere debent, ut sub eundem denominatorem

$$(x-\alpha_1)\,(x-\alpha_2)\,(x-\alpha_3)\,.\,.\,(x-\alpha_n)$$

collectae numeratorem nanciscantur

$$a_0 + a_1\,x + a_2\,x^2 + .\,. + a_{n-k}\,x^{n-k}.$$

Itaque coëfficientum a_0, a_1, a_2, . . , a^{n-k} numerus cum sit $n-k+1$, numeratores illi, quorum est numerus

$$\frac{n\,(n-1)\,(n-2)\,.\,.\,(n-k+1)}{1\,.\,2\,.\,3\,\,.\,.\,\,k},$$

tantum $n-k+1$ conditionibus satisfaciant necesse est. In assignatis igitur istis fractionum simplicium numeratoribus

$$\frac{n\,(n-1)\,(n-2)\,.\,.\,(n-k+1)}{1\,.\,2\,.\,3\,\,.\,.\,\,k} - (n-k+1)$$

quantitates arbitrariae inveniri debent, ut completa esse aestimanda sit problematis resolutio; tot autem reapse in theoremate a nobis proposito inveniuntur. Ecce igitur, dedimus resolutionem problematis generalis completam. —

Sectio III.

Alia quaedam proponuntur, quae ad theoriam fractionum simplicium pertinent.

17.

Hac sectione, nullo ordine observato, alia varia de fractionibus simplicibus afferamus. Proposita fractione

$$\frac{1}{(x-\alpha_1)\,(x-\alpha_2)\,(x-\alpha_3)\,.\,.\,(x-\alpha_n)}$$

in simplices resoluta hasce:

$$\frac{1}{x-\alpha_1}\cdot\frac{1}{(\alpha_1-\alpha_2)\,(\alpha_1-\alpha_3)\,.\,.\,(\alpha_1-\alpha_n)}$$

$$+\frac{1}{x-\alpha_2}\cdot\frac{1}{(\alpha_2-\alpha_1)\,(\alpha_2-\alpha_3)\,.\,.\,(\alpha_2-\alpha_n)}$$

$$+\frac{1}{x-\alpha_3}\cdot\frac{1}{(\alpha_3-\alpha_1)\,(\alpha_3-\alpha_2)\,.\,.\,(\alpha_3-\alpha_n)}$$

. . . .

$$+ \frac{1}{x - \alpha_n} \cdot \frac{1}{(\alpha_n - \alpha_1)(\alpha_n - \alpha_2) \ldots (\alpha_n - \alpha_{n-1})},$$

has ipsas fractiones

$$\frac{1}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots (\alpha_1 - \alpha_n)},$$

$$\frac{1}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3) \ldots (\alpha_2 - \alpha_n)},$$

$$\frac{1}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \ldots (\alpha_3 - \alpha_n)},$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{(\alpha_n - \alpha_1)(\alpha_n - \alpha_2) \ldots (\alpha_n - \alpha_{n-1})}$$

rursus in simplices licet resolvere. Erit e. g.

$$\frac{1}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) \ldots (\alpha_1 - \alpha_n)} =$$

$$\frac{1}{(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4) \ldots (\alpha_2 - \alpha_n)}$$

$$+ \frac{1}{(\alpha_1 - \alpha_3)(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_4) \ldots (\alpha_3 - \alpha_n)}$$

$$+ \frac{1}{(\alpha_1 - \alpha_4)(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3) \ldots (\alpha_4 - \alpha_n)}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{1}{(\alpha_1 - \alpha_n)(\alpha_n - \alpha_2)(\alpha_n - \alpha_3) \ldots (\alpha_n - \alpha_{n-1})}.$$

Quo facto fractiones

$$\frac{1}{(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4) \ldots (\alpha_2 - \alpha_n)},$$

$$\frac{1}{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_4) \ldots (\alpha_3 - \alpha_n)},$$

$$\frac{1}{(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3) \ldots (\alpha_4 - \alpha_n)},$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{(\alpha_n - \alpha_2)(\alpha_n - \alpha_3) \ldots (\alpha_n - \alpha_{n-1})}$$

rursus in simplices licet resolvere, quo repetito negotio atque ad reliquas fractiones consimiles omnes adhibito, tandem devenitur ad formulam sequentem:

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)} = \Sigma \frac{1}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_4)\ldots(\alpha_{n-1}-\alpha_n)}.$$

In quibus charactere Σ rursus complectimur has omnes varias expressiones, quae permutatis elementis $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ seu eorum indicibus 1, 2, 3, ..., n eruuntur.

18.

Sit fx terminus generalis seriei $(n-1)^{ti}$ ordinis, sive

$$fx = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1};$$

datis n terminis seriei, qui indicibus $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ respondent, scilicet $f(\alpha_1)$, $f(\alpha_2)$, $f(\alpha_3), \ldots, f(\alpha_n)$, agitur de ipso determinando termino generali $f(x)$.

Vidimus esse

$$\frac{fx}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)} =$$
$$\frac{f\alpha_1}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)} +$$
$$\frac{f\alpha_2}{(x-\alpha_2)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\ldots(\alpha_2-\alpha_n)} +$$
$$\frac{f\alpha_3}{(x-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\ldots(\alpha_3-\alpha_n)} +$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{f(\alpha_n)}{(x-\alpha_n)(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\ldots(\alpha_n-\alpha_{n-1})}.$$

Qua aequatione multiplicata per

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n),$$

fit $fx =$

$$f\alpha_1 \cdot \frac{(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)\ldots(x-\alpha_n)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)\ldots(\alpha_1-\alpha_n)} +$$
$$f\alpha_2 \cdot \frac{(x-\alpha_1)(x-\alpha_3)(x-\alpha_4)\ldots(x-\alpha_n)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)\ldots(\alpha_2-\alpha_n)} +$$
$$f\alpha_3 \cdot \frac{(x-\alpha_1)(x-\alpha_2)(x-\alpha_4)\ldots(x-\alpha_n)}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)\ldots(\alpha_3-\alpha_n)} +$$
$$\cdots\cdots\cdots\cdots$$
$$f\alpha_n \cdot \frac{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_{n-1})}{(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)(\alpha_n-\alpha_3)\ldots(\alpha_n-\alpha_{n-1})}.$$

Quod alias iam notum, ab Ill. Lagrange propositum, de interpolatione serierum theorema alio modo atque vulgo fit demonstrasse iuvat.

19.

Resolutione in fractiones simplices facta facile etiam hoc demonstratur theorema:

$$\frac{1}{(x-\alpha_1-\beta_1)(x-\alpha_2-\beta_1)\,.\,.\,(x-\alpha_n-\beta_1)(\beta_1-\beta_2)(\beta_1-\beta_3)\,.\,.\,(\beta_1-\beta_m)}$$

$$+\frac{1}{(x-\alpha_1-\beta_2)(x-\alpha_2-\beta_2)\,.\,.\,(x-\alpha_n-\beta_2)(\beta_2-\beta_1)(\beta_2-\beta_3)\,.\,.\,(\beta_2-\beta_m)}$$

$$+\frac{1}{(x-\alpha_1-\beta_3)(x-\alpha_2-\beta_3)\,.\,.\,(x-\alpha_n-\beta_3)(\beta_3-\beta_1)(\beta_3-\beta_2)\,.\,.\,(\beta_3-\beta_m)}$$

. . . .

$$+\frac{1}{(x-\alpha_1-\beta_m)(x-\alpha_2-\beta_m)\,.\,.\,(x-\alpha_n-\beta_m)(\beta_m-\beta_1)(\beta_m-\beta_2)\,.\,.\,(\beta_m-\beta_{m-1})}$$

$$=$$

$$\frac{1}{(x-\alpha_1-\beta_1)(x-\alpha_1-\beta_2)\,.\,.\,(x-\alpha_1-\beta_m)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\,.\,.\,(\alpha_1-\alpha_n)}$$

$$+\frac{1}{(x-\alpha_2-\beta_1)(x-\alpha_2-\beta_2)\,.\,.\,(x-\alpha_2-\beta_m)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\,.\,.\,(\alpha_2-\alpha_n)}$$

$$+\frac{1}{(x-\alpha_3-\beta_1)(x-\alpha_3-\beta_2)\,.\,.\,(x-\alpha_3-\beta_m)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\,.\,.\,(\alpha_3-\alpha_n)}$$

. . . .

$$+\frac{1}{(x-\alpha_n-\beta_1)(x-\alpha_n-\beta_2)\,.\,.\,(x-\alpha_n-\beta_m)(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\,.\,.\,(\alpha_n-\alpha_{n-1})}.$$

20.

Casus memorabilis fractionum simplicium is est, quo earum denominatores seriem arithmeticam effingunt. Sit e. g. $\alpha_1 = p$, $\alpha_2 = p-h$, $\alpha_3 = p-2h$, . ., $\alpha_n = p-(n-1)h$: erit

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\,.\,.\,(\alpha_1-\alpha_n) = \quad 1\,.\,2\,.\,3\,.\,.\,(n-1)\,h^{n-1}$$

$$(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\,.\,.\,(\alpha_2-\alpha_n) = -\,1\,.\,2\,.\,3\,.\,.\,(n-2)\,h^{n-1}$$

$$(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\,.\,.\,(\alpha_3-\alpha_n) = \quad 1\,.\,2\,.\,3\,.\,.\,(n-3)\,1\,.\,2\,.\,h^{n-1}$$

$$(\alpha_4-\alpha_1)(\alpha_4-\alpha_2)\,.\,.\,(\alpha_4-\alpha_n) = -\,1\,.\,2\,.\,3\,.\,.\,(n-4)\,1\,.\,2\,.\,3\,.\,h^{n-1}$$

. . . .

$$(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\,.\,.\,(\alpha_n-\alpha_{n-1}) = (-1)^{n-1}\,1\,.\,2\,.\,3\,.\,.\,(n-1)\,h^{n-1}.$$

Hinc fit, ea suppositione facta,

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\,.\,.\,(x-\alpha_n)} =$$

$$\frac{1}{1.2.3\,.\,.\,(n-1)\,h^{n-1}}\left(\frac{1}{x-\alpha_1}-\frac{n-1}{x-\alpha_2}+\frac{(n-1)(n-2)}{1\,.\,2\,.\,(x-\alpha_3)}-\frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,(x-\alpha_4)}+\text{etc.}\right),$$

sive

$$\frac{1}{x-\alpha_1} - \frac{n-1}{x-\alpha_2} + \frac{(n-1)(n-2)}{1 \cdot 2} \cdot \frac{1}{x-\alpha_3} - \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{x-\alpha_4} + \text{etc}$$

$$= \frac{1 \cdot 2 \cdot 3 \ldots (n-1)\, h^{n-1}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)} .$$

E theoria autem serierum et differentiarum notum est seriei alicuius

$$A_1,\ A_2,\ A_3,\ \ldots,\ A_n,$$

esse $(n-1)^{tam}$ differentiam, quam designemus per $\Delta^{n-1} A_1$, =

$$A_1 - (n-1) A_2 + \frac{(n-1)(n-2)}{1 \cdot 2} A_3 - \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} A_4 + \text{etc.},$$

differentiis ita sumtis, ut a quoque termino is, qui insequitur, detrahatur. Unde videmus, proposita aliqua serie

$$\frac{1}{x-\alpha_1},\ \frac{1}{x-\alpha_2},\ \frac{1}{x-\alpha_3},\ \ldots,\ \frac{1}{x-\alpha_n},$$

elementis α_1, α_2, α_3, ..., α_n effingentibus seriem arithmeticam p, $p-h$, $p-2h$, ..., $p-(n-1)h$, fore $(n-1)^{tam}$ eius differentiam

$$\Delta^{n-1} \frac{1}{x-\alpha_1} = \frac{1}{x-\alpha_1} - \frac{n-1}{x-\alpha_2} + \frac{(n-1)(n-2)}{1 \cdot 2} \cdot \frac{1}{x-\alpha_3} - \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{x-\alpha_4} + \text{etc.}$$

$$= \frac{1 \cdot 2 \cdot 3 \ldots (n-1)\, h^{n-1}}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)} .$$

21.

Iam, ut melius perspiciatur, quomodo fractiones simplices ad theoriam differentiarum et serierum pertineant, haec apponamus.

Seriei alicuius termini, qui respondent indicibus α_1, α_2, α_3, α_4, ..., α_n, sint

$$A_1,\ A_2,\ A_3,\ A_4\ \ldots$$

De hac serie ita deriventur series

$$\begin{array}{l} B_1,\ B_2,\ B_3,\ B_4\ \ldots, \\ C_1,\ C_2,\ C_3,\ C_4\ \ldots, \\ D_1,\ D_2,\ D_3,\ D_4\ \ldots, \\ \text{etc.}, \end{array}$$

ut sit

$$B_1 = \frac{A_1 - A_2}{\alpha_1 - \alpha_2},\quad B_2 = \frac{A_2 - A_3}{\alpha_2 - \alpha_3},\quad B_3 = \frac{A_3 - A_4}{\alpha_3 - \alpha_4},\ \text{etc.}$$

$$C_1 = \frac{B_1 - B_2}{\alpha_1 - \alpha_3},\quad C_2 = \frac{B_2 - B_3}{\alpha_2 - \alpha_4},\quad C_3 = \frac{B_3 - B_4}{\alpha_3 - \alpha_5},\ \text{etc.}$$

$$D_1 = \frac{C_1 - C_2}{\alpha_1 - \alpha_4},\quad D_2 = \frac{C_2 - C_3}{\alpha_2 - \alpha_5},\quad D_3 = \frac{C_3 - C_4}{\alpha_3 - \alpha_6},\ \text{etc.}$$

etc.

Iam patet, indicibus α_1, α_2, α_3, . . effingentibus seriem arithmeticam, p, p—h, p—2h, . . . fore

$$B_1 = \frac{\Delta A_1}{h}, \quad C_1 = \frac{\Delta^2 A_1}{1 \cdot 2 \cdot h^2}, \quad D_1 = \frac{\Delta^3 A_1}{1 \cdot 2 \cdot 3 \cdot h^3}, \text{ etc.}$$

$$B_2 = \frac{\Delta A_2}{h}, \quad C_2 = \frac{\Delta^2 A_2}{1 \cdot 2 \cdot h^2}, \quad D_2 = \frac{\Delta^3 A_2}{1 \cdot 2 \cdot 3 \cdot h^3}, \text{ etc.}$$

$$B_3 = \frac{\Delta A_3}{h}, \quad C_3 = \frac{\Delta^2 A_3}{1 \cdot 2 \cdot h^2}, \quad D_3 = \frac{\Delta^3 A_3}{1 \cdot 2 \cdot 3 \cdot h^3}, \text{ etc.}$$

etc.,

designante $\Delta^n A_m$, ut vulgo fit, n^{tam} differentiam seriei

$$A_m, \quad A_{m+1}, \quad A_{m+2}, \quad A_{m+3}, \quad \text{etc.},$$

differentiis ita sumtis, ut a quoque termino is, qui insequitur, detrahatur.

Seriem B_1, B_2, B_3, B_4, etc. ita quoque exhibere licet:

$$B_1 = \frac{A_1}{\alpha_1 - \alpha_2} + \frac{A_2}{\alpha_2 - \alpha_1}$$

$$B_2 = \frac{A_2}{\alpha_2 - \alpha_3} + \frac{A_3}{\alpha_3 - \alpha_2}$$

$$B_3 = \frac{A_3}{\alpha_3 - \alpha_4} + \frac{A_4}{\alpha_4 - \alpha_3}$$

etc.

Hinc erit

$$C_1 = \frac{A_1}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{A_2}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{A_3}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}$$

$$C_2 = \frac{A_2}{(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)} + \frac{A_3}{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_4)} + \frac{A_4}{(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3)}$$

$$C_3 = \frac{A_3}{(\alpha_3 - \alpha_4)(\alpha_3 - \alpha_5)} + \frac{A_4}{(\alpha_4 - \alpha_3)(\alpha_4 - \alpha_5)} + \frac{A_5}{(\alpha_5 - \alpha_3)(\alpha_5 - \alpha_4)}$$

etc.

Hinc derivatur

$$D_1 = \frac{A_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)} + \frac{A_2}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)} + \frac{A_3}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)} + \frac{A_4}{(\alpha_4-\alpha_1)(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)}$$

$$D_2 = \frac{A_2}{(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_2-\alpha_5)} + \frac{A_3}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)(\alpha_3-\alpha_5)} + \frac{A_4}{(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)(\alpha_4-\alpha_5)} + \frac{A_5}{(\alpha_5-\alpha_2)(\alpha_5-\alpha_3)(\alpha_5-\alpha_4)}$$

$$D_3 = \frac{A_3}{(\alpha_3-\alpha_4)(\alpha_3-\alpha_5)(\alpha_3-\alpha_6)} + \frac{A_4}{(\alpha_4-\alpha_3)(\alpha_4-\alpha_5)(\alpha_4-\alpha_6)} + \frac{A_5}{(\alpha_5-\alpha_3)(\alpha_5-\alpha_4)(\alpha_5-\alpha_6)} + \frac{A_6}{(\alpha_6-\alpha_3)(\alpha_6-\alpha_4)(\alpha_6-\alpha_5)}.$$

etc.

Unde videmus, ubi erit

$$A_1 = \frac{1}{x - \alpha_1}, \quad A_2 = \frac{1}{x - \alpha_2}, \quad A_3 = \frac{1}{x - \alpha_3}, \quad A_4 = \frac{1}{x - \alpha_4}, \text{ etc.},$$

fore

fore:

$$B_1 = \frac{1}{(x-\alpha_1)(x-\alpha_2)}, \quad B_2 = \frac{1}{(x-\alpha_2)(x-\alpha_3)}, \quad B_3 = \frac{1}{(x-\alpha_3)(x-\alpha_4)}, \quad \text{etc.};$$

$$C_1 = \frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)}, \; C_2 = \frac{1}{(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)} \; C_3 = \frac{1}{(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)}, \; \text{etc.};$$

$$D_1 = \frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)}, D_2 = \frac{1}{(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)}, D_3 = \frac{1}{(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)(x-\alpha_6)}, \text{etc.};$$

etc.

Hinc statim patet, ubi $\alpha_1 = p$, $\alpha_2 = p - h$, $\alpha_3 = p - 2h$, etc., fore

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)\,.\,.\,(x-\alpha_n)} = \frac{\Delta^{n-1}\frac{1}{x-\alpha_1}}{1\,.\,2\,.\,3\,.\,.\,(n-1)\,h^{n-1}}.$$

Sicuti a differentiis ad terminos seriei primariae, ita quoque a terminis A_1, B_1, C_1, D_1 etc. ad terminos seriei propositae A_1, A_2, A_3, A_4, etc. ascendere licet. Facto periculo, nasci videmus elegantissimas formulas:

$$A_1 = A_1$$
$$A_2 = A_1 + (\alpha_2 - \alpha_1) B_1$$
$$A_3 = A_1 + (\alpha_3 - \alpha_1) B_1 + (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) C_1$$
$$A_4 = A_1 + (\alpha_4 - \alpha_1) B_1 + (\alpha_4 - \alpha_1)(\alpha_4 - \alpha_2) C_1 + (\alpha_4 - \alpha_1)(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3) D_1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$A_n = A_1 + (\alpha_n - \alpha_1) B_1 + (\alpha_n - \alpha_1)(\alpha_n - \alpha_2) C_1 + (\alpha_n - \alpha_1)(\alpha_n - \alpha_2)(\alpha_n - \alpha_3) D_1 + \text{etc.}$$

Haec alias iam nota*) adnotasse sufficiat; iniuria autem negliguntur ab Analystis, quae multo sunt generaliora theoremata iis, quae vulgo de seriebus ac differentiis circumferuntur.

S e c t i o IV.

Theoremata de singulari quadam serierum infinitarum transformatione.

§. 22.

Proposita serie infinita S, cuius terminus generalis sive x^{tus} est

$$\frac{p\,(p+1)\,(p+2)\,.\,.\,(p+x-1)}{1\,.\,2\,.\,3\,.\,.\,x} \cdot \frac{(a_0 + a_1 x + a_2 x^2 + \,.\,.\, + a_{n-1} x^{n-1})\, y^x}{(x+\alpha_1)\,(x+\alpha_2)\,.\,.\,.\,(x+\alpha_n)},$$

transformatis singulis eius terminis per methodum fractionum simplicium, n series nanciscimur, quarum termini generales, reiectis factoribus constantibus, sunt

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*) v. Principia Phil. Nat. ed. Amstelod. a. 1723. lib. III. lemma 8. pag. 446.

$$\frac{p(p+1)(p+2)\ldots(p+x-1)}{1.2.3\ldots x}\cdot\frac{y^x}{x+\alpha_1},$$
$$\frac{p(p+1)(p+2)\ldots(p+x-1)}{1.2.3\ldots x}\cdot\frac{y^x}{x+\alpha_2},$$
$$\frac{p(p+1)(p+2)\ldots(p+x-1)}{1.2.3\ldots x}\cdot\frac{y^x}{x+\alpha_3};$$
$$\cdots\cdots\cdots$$
$$\frac{p(p+1)(p+2)\ldots(p+x-1)}{1.2.3\ldots x}\cdot\frac{y^x}{x+\alpha_n}.$$

Iam erit series, cuius terminus generalis est

$$\frac{p(p+1)(p+2)\ldots(p+x-1)}{1.2.3\ldots x}\cdot\frac{y^x}{x+a},$$

loco x positis omnibus numeris integris inde a 0 usque ad ∞:

$$\frac{1}{a}+\frac{p.y}{a+1}+\frac{p(p+1)}{1.2}\cdot\frac{y^2}{a+2}+\frac{p(p+1)(p+2)}{1.2.3}\cdot\frac{y^3}{a+3}+\text{etc.}$$
$$=\frac{1}{y^a}\int\frac{y^{a-1}dy}{(1-y)^p}.$$

Hinc posito

$$a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}=fx,$$
$$(x+\alpha_1)(x+\alpha_2)(x+\alpha_3)\ldots(x+\alpha_n)=\varphi x,$$

fit S =

$$\frac{f(-\alpha_1)}{y^{\alpha_1}\varphi'(-\alpha_1)}\int\frac{y^{\alpha_1-1}dy}{(1-y)^p}+\frac{f(-\alpha_2)}{y^{\alpha_2}\varphi'(-\alpha_2)}\int\frac{y^{\alpha_2-1}dy}{(1-y)^p}+\ldots+\frac{f(-\alpha_n)}{y^{\alpha_n}\varphi'(-\alpha_n)}\int\frac{y^{\alpha_n-1}dy}{(1-y)^p}.$$

Integrale $\int\frac{y^{a-1}dy}{(1-y)^p}$ finitum obtineri potest, si aut a, aut p, aut a — p numerus erit integer. Hinc etiam seriei S summa finita assignari potest, ubi aut p, aut a, aut a — p numerus erit integer, loco a singulis positis elementis

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_n.$$

Ubi quantitatum

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_n$$

aliquot aequales existunt, methodo fractionum simplicium huiusmodi series eruimus:

$$\frac{1}{a^m}+\frac{p.y}{(a+1)^m}+\frac{p(p+1)}{1.2}\cdot\frac{y^2}{(a+2)^m}+\frac{p(p+1)(p+2)}{1.2.3}\cdot\frac{y^3}{(a+3)^m}+\text{etc.},$$

cuiusmodi serierum summam, non vulgo annotari solet, statim e seriei

$$\frac{1}{a}+\frac{p.y}{a+1}+\frac{p(p+1)}{1.2}\cdot\frac{y^2}{a+2}+\frac{p(p+1)(p+2)}{1.2.3}\cdot\frac{y^3}{a+3}+\text{tce.}$$

deduci posse. Hac enim posita = s, fit illa $\frac{(-1)^{m-1} d^m s}{1.2.3..(m-1)\, da^{m-1}}$. Qua de re iste nos nihil morabitur casus.

Similiter in libris, qui de elementis calculi integralis agunt, annotari non solet, ex invento integrali

$$\int \frac{(a+bx)\,dx}{c+ex+fx^2}$$

statim sequi

$$\int \frac{(a+bx)\,dx}{(c+ex+fx^2)^m} = \frac{(-1)^{m-1} d^{m-1}}{1.2..(m-1)\, dc^{m-1}} \int \frac{(a+bx)\,dx}{c+ex+fx^2},$$

quae latius etiam patet methodus.

23.

Proposita serie $S_1 =$

$$\frac{1}{a} + \frac{x(a+c)}{a(a+1)} + \frac{x^2(a+c)(a+c+1)}{a(a+1)(a+2)} + \frac{x^3(a+c)(a+c+1)(a+c+2)}{a(a+1)(a+2)(a+3)} + \frac{x^4(a+c)(a+c+1)(a+c+2)(a+c+3)}{a(a+1)(a+2)(a+3)(a+4)} + \text{etc.},$$

omnes eius termini in fractiones simplices resolvantur; ita ut sit:

$$\frac{1}{a} = \frac{1}{a}$$

$$\frac{x(a+c)}{a(a+1)} = \frac{xc}{a.1.} - \frac{x(c-1)}{a+1}$$

$$\frac{x^2(a+c)(a+c+1)}{a(a+1)(a+2)} = \frac{x^2 c(c+1)}{a.1.2} - \frac{x^2(c-1)c}{(a+1)1.} + \frac{x^2(c-2)(c-1)}{(a+2)1.2}$$

$$\frac{x^3(a+c)(a+c+1)(a+c+2)}{a(a+1)(a+2)(a+3)} = \frac{x^3 c(c+1)(c+2)}{a.1.2.3} - \frac{x^3(c-1)c(c+1)}{(a+1)1.2.1} + \frac{x^3(c-2)(c-1)c}{(a+2)1.1.2} - \frac{x^3(c-3)(c-2)(c-1)}{(a+3)1.2.3}$$

$$\frac{x^4(a+c)(a+c+1)(a+c+2)(a+c+3)}{a(a+1)(a+2)(a+3)(a+4)} = \frac{x^4 c(c+1)(c+2)(c+3)}{a\,1.2.3.4} - \frac{x^4(c-1)c(c+1)(c+2)}{(a+1)1.2.3.1} + \frac{x^4(c-2)(c-1)c(c+1)}{(a+2)1.2.1.2} - \frac{x^4(c-3)(c-2)(c-1)c}{(a+3)1.1.2.3}$$
$$+ \frac{x^5(c-4)(c-3)(c-2)(c-1)}{(a+4)1.2.3.4}$$

etc. etc.

Hic omnes series verticales continere factorem videmus

$$\frac{1}{(1-x)^c} = 1 + cx + \frac{c(c+1)}{1.2}x^2 + \frac{c(c+1)(c+2)}{1.2.3}x^3 + \text{etc.};$$

quo collecto fit S =

$$\left[\frac{1}{a} - \frac{x(c-1)}{a+1} + \frac{x^2(c-1)(c-2)}{(a+2)1.2} - \frac{x^3(c-1)(c-2)(c-3)}{(a+3)1.2.3} + \text{etc.}\right]\frac{1}{(1-x)^c}$$

$$= \frac{1}{(1-x)^c x^a}\int x^{a-1}(1-x)^{c-1}\,dx.$$

Summam S igitur videmus finitam assignari posse, ubi aut a aut c aut a + c numerus integer erit. Tum enim integrale

$$\int x^{a-1}(1-x)^{c-1}\,dx$$

finitum exhiberi potest.

Posito $c = \infty$, $x = \frac{y}{c} = \frac{y}{\infty}$, fit

$$(a + c)(a + c + 1)(a + c + 2) \ldots (a + c + m) = c^{m+1};$$

$$(c - 1)(c - 2)(c - 3)(c - m) = c^m;$$

$$(1 - x)^c = (1 - \frac{y}{\infty})^\infty = e^{-y}, \text{ unde } \frac{1}{(1 - x)^c} = e^{+y}.\text{*)}$$

His substitutis fit $S =$

$$\frac{1}{a} + \frac{y}{a(a+1)} + \frac{y^2}{a(a+1)(a+2)} + \frac{y^3}{a(a+1)(a+2)(a+3)} + \text{etc.} =$$

$$\left(\frac{1}{a} - \frac{y}{a+1} + \frac{y^2}{1.2(a+2)} - \frac{y^3}{1.2.3(a+3)} + \text{etc.}\right) e^y,$$

quam transformationem etiam directa via deducere licuit.

24.

Dedimus Sect. III. §. 4. formulam

$$\frac{1}{(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)} = \frac{1}{1.2.3 \ldots (n-1) h^{n-1}} \Delta^{n-1} \frac{1}{x - \alpha_1},$$

posito $\alpha_1 = p$, $\alpha_2 = p - h$, $\alpha_3 = p - 2h$, ..., $\alpha_n = p - (n-1)h$. Ponatur $x - \alpha_1 = a$, $h = -1$, unde

$$\alpha_2 = a + 1,\ \alpha_3 = a + 2,\ \alpha_4 = a + 3, \ldots, \alpha_n = a + n - 1.$$

His substitutis atque loco n posito $n + 1$ e formula apposita videmus, seriei

$$\frac{1}{a}, \frac{1}{a+1}, \frac{1}{a+2}, \frac{1}{a+3}, \text{ etc.}$$

esse n^{tam} differentiam sive $\Delta^n \frac{1}{a} = \frac{(-1)^n\, 1.2.3 \ldots n}{a(a+1)(a+2) \ldots (a+n)}$; differentiis ita sumtis, ut a quovis termino is, qui insequitur, detrahatur; sive differentiis ita sumtis, ut quivis terminus ab insequente detrahatur,

$$\Delta^n \frac{1}{a} = \frac{1.2.3 \ldots n}{a(a+1)(a+2) \ldots (a+n)},$$

unde

$$\frac{\Delta^n \frac{1}{a}}{1.2.3 \ldots n} = \frac{1}{a(a+1)(a+2) \ldots (a+n)}.$$

Ex hac aequatione

$$\frac{1}{a(a+1)(a+2) \ldots (a+n)} = \frac{\Delta^n \frac{1}{a}}{1.2.3 \ldots n},$$

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*) conf. Eul. Introd. in Analys. Infin. lib. I. cap. VII. §. 115 seq.

sequitur $\Delta^p \frac{1}{a(a+1)(a+2)\,..\,(a+n)} = \frac{\Delta^{n+p}\frac{1}{a}}{1\,.\,2\,.\,3\,..\,n}$

$$= \frac{1\,.\,2\,.\,3\,..\,(n+p)}{1\,.\,2\,.\,3\,..\,n} \cdot \frac{1}{a(a+1)(a+2)\,..\,(a+n+p)} = \frac{(n+1)(n+2)\,..\,(n+p)}{a(a+1)(a+2)\,..\,(a+n+p)}.$$

Huius theorematis ope sumatur p^{ta} differentia seriei S =

$$\frac{1}{a} + \frac{y}{a(a+1)} + \frac{y^2}{a(a+1)(a+2)} + \frac{y^3}{a(a+1)(a+2)(a+3)} + \text{etc.} =$$

$$\left(\frac{1}{a} - \frac{y}{a+1} + \frac{y^2}{1\,.\,2\,(a+2)} - \frac{y^3}{1\,.\,2\,.\,3\,(a+3)} + \text{etc.}\right) e^y,$$

respectu elementi a habito. Fit $\Delta^p S$ =

$$\frac{1\,.\,2\,.\,3\,..\,p}{a(a+1)\,..\,(a+p)} + \frac{2\,.\,3\,.\,4\,..\,(p+1)\,y}{a(a+1)\,..\,(a+p+1)} + \frac{3\,.\,4\,.\,5\,..\,(p+2)\,y^2}{a(a+1)\,..\,(a+p+2)} + \text{etc.}$$

$$= \left[\frac{1\,.\,2\,.\,3\,..\,p}{a(a+1)\,..\,(a+p)} - \frac{1\,.\,2\,.\,3\,..\,p\,.\,y}{(a+1)(a+2)\,..\,(a+p+1)} + \frac{1\,.\,2\,.\,3\,..\,p\,.\,y^2}{(a+2)(a+3)\,..\,(a+p+2)}\right] e^y;$$

sive divisione facta per factorem communem

$$\frac{1\,.\,2\,.\,3\,..\,p}{a(a+1)\,..\,(a+p)},$$

$$1+(p+1)\frac{y}{a+p+1} + \frac{(p+1)(p+2)}{1\,.\,2}\cdot\frac{y^2}{(a+p+1)(a+p+2)} + \frac{(p+1)(p+2)(p+3)}{1\,.\,2\,.\,3}\cdot\frac{y^3}{(a+p+1)(a+p+2)(a+p+3)} + \text{etc.}$$

$$= \left[1 - \frac{a}{(a+p+1)}y + \frac{1}{1\,.\,2}\cdot\frac{a(a+1)}{(a+p+1)(a+p+2)}y^2 - \frac{1}{1\,.\,2\,.\,3}\cdot\frac{a(a+1)(a+2)}{(a+p+1)(a+p+2)(a+p+3)} + \text{etc.}\right] e^y.$$

25.

At multo generalius transformationis genus hac methodo invenire licet. Ex aequatione

$$\frac{x^p}{(1-x)^c} = x^p + cx^{p+1} + \frac{c(c+1)}{1\,.\,2}x^{p+2} + \frac{c(c+1)(c+2)}{1\,.\,2\,.\,3}x^{p+3} + \text{etc.},$$

sequitur $d^n \frac{x^p}{(1-x)^c}$ =

$$p(p-1)\,..\,(p-n+1)\,x^{p-n}\left[1 + \frac{p+1}{p-n+1}\cdot cx + \frac{(p+1)(p+2)}{(p-n+1)(p-n+2)}\cdot\frac{c(c+1)}{1\,.\,2}x^2 + \,..\right] dx^n;$$

unde posito S =

$$1 + \frac{p+1}{p-n+1}cx + \frac{(p+1)(p+2)}{(p-n+1)(p-n+2)}\cdot\frac{c(c+1)}{1\,.\,2}x^2 + \text{etc.},$$

fit $S = \frac{1}{p(p-1)\,..\,(p-n+1)\,x^{p-n}\,dx^n}\, d^n \frac{x^p}{(1-x)^c}$.

Iam ex notatione nostra Sect. I. §. 8. indicata est

$$d^n \frac{x^p}{(1-x)^c\, dx^n} = 1 \cdot 2 \ldots n \left[\frac{(x+h)^p}{(1-x-h)^c}\right]_{h^n}$$

Iam est

$$\frac{(x+h)^p}{(1-x-h)^c} = \frac{1}{(1-x)^{c-p}} \cdot \frac{\left(\frac{x}{1-x} + \frac{h}{1-x}\right)^p}{\left(1 - \frac{h}{1-x}\right)^c};$$

unde etiam

$$\left[\frac{(x+h)^p}{(1-x-h)^c}\right]_{h^n} = \frac{1}{(1-x)^{c-p}} \left\{\frac{\left(\frac{x}{1-x} + \frac{h}{1-x}\right)^p}{\left(1 - \frac{h}{1-x}\right)^c}\right\}_{h^n}.$$

Ex aequatione autem

$$\left[F(uh)\right]_{h^n} = u^n \left[F(h)\right]_{h^n},$$

quae demonstratione non eget, sequitur

$$\left\{\frac{\left(\frac{x}{1-x} + \frac{h}{1-x}\right)^p}{\left(1 - \frac{h}{1-x}\right)^c}\right\}_{h^n} = \frac{1}{(1-x)^n} \left\{\frac{\left(\frac{x}{1-x} + h\right)^p}{(1-h)^c}\right\}_{h^n}.$$

Est autem $\left\{\frac{\left(\frac{x}{1-x} + h\right)^p}{(1-h)^c}\right\}_{h^n} = \frac{1}{(1-x)^p} \left[\frac{(h+x\,(1-h))^p}{(1-h)^c}\right]_{h^n}$,

unde tandem colligimus

$$\left[\frac{(x+h)^p}{(1-x-h)^c}\right]_{h^n} = \frac{1}{(1-x)^{c+n}} \left[\frac{(h+x\,(1-h))^p}{(1-h)^c}\right]_{h^n},$$

quod theorema, sicuti methodum, qua eo perventum est, attentione Analystarum dignum puto. Eo enim clarissimae seriei hypergeometricae insignis transformatio nititur.

Contemplemur enim expressionem, ad quam devenimus,

$$\left[\frac{(h+x\,(1-h))^p}{(1-h)^c}\right]_{h^n}.$$

Evoluta expressione

$$\frac{(h+x\,(1-h))^p}{(1-h)^c},$$

atque reiectis iis, qui terminum h^n continere non possunt, terminis, positoque $p-n-c=a$, nanciscimur

$$\frac{p(p-1)\ldots(p-n+1)}{1 \cdot 2 \ldots n} h^n x^{p-n} (1-h)^a + \frac{p(p-1)\ldots(p-n)}{1 \cdot 2 \ldots (n+1)} h^{n-1} x^{p-n+1} (1-h)^{a+1} +$$

p (p—1)

$$\frac{p(p-1)\,.\,.\,(p-n-1)}{1\,.\,2\,.\,.\,(n+2)}\,h^{n-2}\,x^{p-n+2}\,(1-h)^{a+2}\,+\,\text{etc.}$$

$$=\frac{p(p-1)\,.\,.\,(p-n+1)}{1\,.\,2\,.\,.\,n}\,x^{p-n}\left[h^n(1-h)^a+\frac{p-n}{n+1}\,x\,h^{n-}(1-h)^{a+1}+\frac{(p-n)\,(p-n-1)}{(n+1)\,(n+2)}x^2h^{n-2}(1-h)^{a+2}+\text{etc.}\right],$$

unde

$$\frac{1\,.\,2\,.\,.\,n}{p(p-1)\,.\,.\,(p-n+1)\,x^{p-n}}\,.\,\left[\frac{(h+x\,(1-h))^p}{(1-h)^c}\right]_{h^n}=$$

$$\left[(1-h)^a\right]_{h^0}+\frac{p-n}{n+1}\,.\,\left[(1-h)^{a+1}\right]_h+\frac{(p-n)\,(p-n-1)}{(n+1)\,(n+2)}\left[(1-h)^{a+2}\right]_{h^2}+\text{etc.}$$

$$=1-\frac{p-n}{n+1}\,.\,\frac{(a+1)}{1}x+\frac{(p-n)\,(p-n-1)}{(n+1)\,(n+2)}\,.\,\frac{(a+1)\,(a+2)}{1\,.\,2}\,x^2+\text{etc.}$$

Hinc fit $S=\dfrac{1}{p(p-1)\,.\,.\,(p-n+1)\,x^{p-n}\,dx^n}\,d^n\,\dfrac{x^p}{(1-x)^c}$

$$=\frac{1\,.\,2\,.\,3\,.\,.\,n}{p(p-1)\,.\,.\,(p-n+1)\,x^{p-n}}\,.\,\left[\frac{(x+h)^p}{(1-x-h)^c}\right]_{h^n}$$

$$=\frac{1\,.\,2\,.\,3\,.\,.\,n}{p\,(p-1)\,.\,.\,(p-n+1)\,x^{p-n}}\,.\,\frac{1}{(1-x)^{c+n}}\left[\frac{(h+x\,(1-h))^p}{(1-h)^c}\right]_{h^n}=$$

$$\left[1-\frac{p-n}{n+1}\,.\,\frac{a+1}{1}\,x+\frac{(p-n)\,(p-n-1)}{(n+1)\,(n+2)}\,.\,\frac{(a+1)\,(a+2)}{1\,.\,2}\,x^2+\text{etc.}\right]\frac{1}{(1-x)^{c+n}}\,.$$

Huius theorematis casus speciales tantum sunt, quos hactenus tractavimus.

Dedit hanc transformationem primus Ill. Euler in commentatione inscripta:

Specimen transformationis singularis serierum,

quae legitur in Nov. Act. Academ. Petrop. tom. XII. pag. 58—70. Quam commentationem etsi autor iam anno 1778 conventui exhibuisset, tamen in tomo XII. demum Novorum Actorum legitur, qui anno 1801 lucem vidit. Hinc factum est, ut Ill. Pfaff in *Disquisitionibus Analyticis*, quae anno 1797 prodierunt, primus eam exhibuisse sibi visus sit.*) Harum enim altera disquisitio est de integratione aequationis differentio-differentialis:

$$x^2\,(a+bx^n)\,d^2y+x\,(c+ex^n)\,dy\,dx+(f+gx^n)\,y\,dx^2=X\,dx^2,$$

a cuius integratione seriei nostrae S summatio pendet. In qua integratione occupati et Euler et Pfaff, eandem fere viam secuti, ad hanc nostram seriei S pervenerunt transformationem. Mox fusius de hac transformatione egit Ill. Pfaff in doctissima commentatione inscripta:

Observationes analyticae ad L. Euleri Institutiones Calculi Integralis
Vol. IV. Supplem. II. et IV.,

quae Academiae Petropol. ab autore tradita a. 1797, in tomo XI. legitur Novorum Actorum (Histoire pag. 37, Supplément), qui anno 1798 prodiit. Namque in illo Vol. IV. Calculi Integralis (pag. 245) tum recens edito haec transformatio apposita tamquam lemma

*) v. Disquis. II. §. XXVIII. pag. 160. 161.

nec addita demonstratione legebatur. Unde in Observationibus laudatis Ill. Pfaff occasionem cepit, hanc retractandi transformationem variasque eius demonstrationes adstruendi, ingeniosas, ut ille solebat. Nec nostra fortasse demonstratio Analystis displicebit; quae aequatione nititur

$$\left[\frac{(x+h)^p}{(1-x-h)^c}\right]_{h^n} = \frac{1}{(1-x)^{c+n}}\left[\frac{(h+x(1-h))^p}{(1-h)^c}\right]_{h^n},$$

cuius utraque pars evoluta, altera seriem S, altera eius transformatam praebet. Haec ipsa vero aequatio statim prodibat e theoremate:

$$\left[F(uh)\right]_{h^n} = u^n\left[F(h)\right]_{h^n}.$$

Hanc methodum peritus cognoverit latius patere, immo ad series hypergeometricas omnium ordinum extendi posse. Ceterum, qui post Ill. Pfaff hanc attigerit transformationem, neminem scio praeter Ill. Gauss, cuius ea de re commentatio in omnium manibus est.

BEROLINI, FORMIS HAYNIANIS.